On the Existence of Symmetric Mixed Strategy Equilibria

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Abstract

In this paper we provide a result ensuring the existence of symmetric mixed strategy equilibria in symmetric games. We apply the fixed point theorem of Glicksberg and Fan analogously to the way in which Moulin (1986, "Game Theory for the Social Sciences", New York) uses Kakutani's theorem to prove the existence of a symmetric equilibrium in pure strategies.

Key words: symmetric equilibria, mixed strategies, existence

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1 Introduction

The famous fixed point theorem of Glicksberg (1952) and Fan (1952) is often used in economic applications to guarantee the existence of mixed strategy Nash equilibria in games in which the (pure) strategy spaces are compact and convex, and the payoff functions are continuous. [2]

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[1] The authors would like to thank Professor Michael Leinert for providing the main idea for the proof of proposition 2, as well as Jürgen Eichberger and Hartmut Kogelschatz for their support.

[2] Mixed strategy equilibria are primarily discussed when the existence of equilibria in pure strategies cannot be guaranteed, mostly due to non-convexities in the best response correspondences. The examples are numerous and include auctions, tournaments, political contests, rent seeking, research and development races, models of imperfect competition, to mention but a few of them. Mixed strategy equilibria of,

In this note we show that *symmetric* games satisfying these premises always possess a *symmetric* equilibrium in mixed strategies. We utilize Glicksberg's (1952) fixed point theorem analogously to the way in which Moulin (1986, pp. 115–116) uses Kakutani's (1941) theorem to prove the existence of a symmetric equilibrium in pure strategies.

The result is useful for the analysis of games in which symmetric equilibria cannot be derived explicitly, and only some of their properties can be established. [3]

2 Setting and Result

We consider a symmetric n-person game: Each player has the same (pure) strategy space A, which is assumed to be compact and Hausdorff [4]. Let the game be given in normal form by the continuous payoff function

$$R\colon A^n \to \mathbb{R},$$

where $R(x, y_1, \ldots, y_{n-1})$ is the payoff of an agent playing x, when the other $n-1$ agents play $y_1, \ldots, y_{n-1}$, respectively [5]. By S we denote the set of all mixed strategies over A, i.e. the set of all regular probability measures on A. From now on, S will be equipped with the weak topology. This implies that S is compact, since that is the case for A (see, for example, Bauer, 1992, p. 237).

The game's mixed strategy extension is described by the expected payoff func-

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for example, the all-pay auction have been studied extensively, as these models do usually not possess pure strategy equilibria, but have many applications (see Baye et al., 1996; Hillman and Riley, 1989).

[3] Burguet and Sakovics (1999), for example, study competition among auctioneers in setting the reserve price in second price auctions. As a symmetric mixed strategy equilibrium cannot be derived formally, they provide a discrete approximation. Damianov and Becker (2005) compare the supports of the symmetric mixed strategy equilibria of the uniform price and the discriminatory auction in a variable supply multi-unit auction model. The present theorem guarantees the existence of symmetric equilibria in these models.

[4] Here we take the same general framework as Glicksberg; an applied economist will probably think of A as a compact subset of the Euclidean space $\mathbb{R}^m$. The theorem, however, applies to much more general cases, such as when strategies are whole supply or demand functions.

[5] As the game is symmetric, $R(x, y_1, \ldots, y_{n-1})$ remains unchanged for all permutations of $y_1, \ldots, y_{n-1}$.

tion $\mathfrak{R}\colon S^n \to \mathbb{R}$, where

$$\mathfrak{R}(\sigma, \tau_1, \ldots, \tau_{n-1}) = \int \ldots \int \int R(x, y_1, \ldots, y_{n-1}) \, \mathrm{d}\sigma(x)\mathrm{d}\tau_1(y_1) \ldots \mathrm{d}\tau_{n-1}(y_{n-1})$$

is the expected payoff of an agent playing the mixed strategy $\sigma \in S$, the others playing $\tau_1, \ldots, \tau_{n-1}$, respectively.

The next theorem states our result.

Theorem 1 (Existence). *A symmetric game, as defined above, has a symmetric Nash equilibrium in mixed strategies.*

PROOF. For the proof we refer to propositions 2, 3 and 4 in the Appendix. Proposition 2 states that $\mathfrak{R}$ is continuous. Let us define the correspondence $\Phi_{\mathfrak{R}}\colon S \to \mathscr{P}(S)$ by

$$\Phi_{\mathfrak{R}}(\tau) := \arg\max_{\sigma \in S} \ \mathfrak{R}(\sigma, \tau, \ldots, \tau);$$

now $\Phi_{\mathfrak{R}}(\tau)$ is the best response set of an agent when all other agents play the same mixed strategy τ.

As can be seen by applying propositions 3 and 4, the space S and the correspondence $\Phi_{\mathfrak{R}}$ satisfy the premises of Glicksberg's (1952) and Fan's (1952) fixed point theorem:

Theorem (Glicksberg, Fan). *Let S be a compact convex subset of a convex Hausdorff linear topological space. Then every upper semi-continuous, convex-valued correspondence $\Phi\colon S \to \mathscr{P}(S)$ with $\Phi(s) \neq \emptyset$ for all $s \in S$ has a fixed point.*

Applying this theorem, we now obtain a fixed point: The theorem ensures the existence of a mixed strategy $\tau^* \in S$ with

$$\tau^* \in \Phi_{\mathfrak{R}}(\tau^*) = \arg\max_{\sigma \in S} \ \mathfrak{R}(\sigma, \tau^*, \ldots, \tau^*),$$

which means, $(\tau^*, \ldots, \tau^*)$ constitutes a symmetric mixed strategy equilibrium.

3 Conclusion

The present paper shows how the existence proof for symmetric pure strategy equilibria carries over to the mixed strategy case. Though straightforward mathematically, the theorem supplements the known existence results. For a related statement in a different framework, with discontinuous payoff functions but a less general strategy space, we refer to Dasgupta and Maskin (1986, theorem 6).

Appendix

Proposition 2. *Let A be a compact Hausdorff space and $R\colon A^n \to \mathbb{R}$ a continuous function. Then $\mathfrak{R}\colon S^n \to \mathbb{R}$, $(\mu^{(1)}, \dots, \mu^{(n)}) \mapsto \int \dots \int R \,\mathrm{d}\mu^{(n)} \dots \mathrm{d}\mu^{(1)}$, is continuous.*

Proof. We show that $\mathfrak{R}$ can be uniformly approximated with arbitrary precision by continuous functions and is, therefore, continuous. Take any $\varepsilon > 0$. The Stone-Weierstraß theorem (see, for example, von Querenburg, 2001, Satz 9.7) ensures the existence of continuous functions $f_\ell^{(j)}\colon A \to \mathbb{R}$, $j = 1, \dots, n$, $\ell = 1, \dots, L$, with

$$\left| R(x_1, \dots, x_n) - \sum_\ell f_\ell^{(1)}(x_1) \cdot \ldots \cdot f_\ell^{(n)}(x_n) \right| < \varepsilon \quad \text{for all } (x_1, \dots, x_n) \in A^n.$$

This implies

$$\left| \mathfrak{R}(\mu^{(1)}, \dots \mu^{(n)}) - \sum_\ell \int f_\ell^{(1)} \mathrm{d}\mu^{(1)} \cdot \ldots \cdot \int f_\ell^{(n)} \mathrm{d}\mu^{(n)} \right| < \varepsilon$$

for all $(\mu^{(1)}, \dots, \mu^{(n)}) \in S^n$. By definition of weak convergence, all of the functions

$$\mu^{(i)} \mapsto \int f_l^{(i)} \mathrm{d}\mu^{(i)}$$

are continuous. So, as a result, is the subtrahend in the above inequality, which gives the desired approximation. □

Proposition 3. *Let S, T be topological spaces. Let S be compact, and the function $F\colon S \times T \to \mathbb{R}$ be continuous. The correspondence $\Phi_F\colon T \to \mathscr{P}(S)$, where $T \mapsto \arg\max_{s \in S} F(s,t)$, is (a) everywhere nonempty and (b) upper semi-continuous.*

Proof. (a) holds, because continuous functions defined on compact sets somewhere take a maximum. To prove *(b),* we show that the set

$$G := \left\{ (s,t) \;\middle|\; t \in T, s \in \Phi_F(t) \right\}$$

(the "graph" of Φ_F) is closed in $S \times T$ or – equivalently – that its complement is open. Take an arbitrary point $(s_0, t_0) \notin G$ and any $\overline{s} \in \Phi_F(t_0)$. Let $m_0 :=$

$F(s_0, t_0)$, $\overline{m} := F(\overline{s}, t_0)$, and $\varepsilon := (\overline{m} - m_0)/3$. As F is continuous, and pre-images of open subsets under continuous functions are open, neighborhoods $U_0 \subseteq S$ of s_0 and $V_0 \subseteq T$ of t_0 exist such that $U_0 \times V_0 \subseteq F^{-1}\Big((-\infty; m_0 + \varepsilon)\Big)$ and $\{\overline{s}\} \times V_0 \subseteq F^{-1}\Big((\overline{m} - \varepsilon; \infty)\Big)$. For each $(s, t) \in U_0 \times V_0$, the inequality $F(\overline{s}, t) > \overline{m} - \varepsilon > m_0 + \varepsilon > F(s, t)$ holds, and, as a result, $(U_0 \times V_0) \cap G = \emptyset$. $\square$

Proposition 4. *Let $f \colon S \to \mathbb{R}$ be a linear function on a convex subset S of a linear space. Then the set $M := \arg\max_{s \in S} f(s)$ is convex.*

Proof. Take any $s_1, s_2 \in S$. Then, for any $\lambda \in (0, 1)$, we have $\lambda s_1 + (1 - \lambda)s_2 \in S$ and $f\Big(\lambda s_1 + (1 - \lambda)s_2\Big) = \lambda f(s_1) + (1 - \lambda)f(s_2) = f(s_1) = f(s_2)$, so that $\lambda s_1 + (1 - \lambda)s_2 \in M$. $\square$

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