Auctions with Variable Supply: Uniform Price versus Discriminatory

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Abstract

We examine an auction in which the seller determines the supply after observing the bids. We compare the uniform price and the discriminatory auction in a setting of supply uncertainty, where uncertainty is caused by the interplay of two factors: the seller's private information about marginal cost and the seller's incentive to sell the profit-maximizing quantity, given the received bids. In every symmetric mixed strategy equilibrium, bidders submit higher bids in the uniform price auction than in the discriminatory auction. In the two-bidder case, this result extends to the set of rationalizable strategies. As a consequence, we find that the uniform price auction generates a higher expected revenue for the seller and a higher trade volume.

Key Words: sealed bid multi-unit auctions, variable supply auctions, discriminatory and uniform price auctions, subgame perfect equilibria, rationalizable strategies.
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1 Introduction

In a variable supply auction the seller determines the quantity to be sold after observing the bids. Such auctions are applied in a variety of markets, e.g. markets for electricity, emission permits and initial public offerings (IPOs), yet they are most common in Treasury markets. For instance, Nyborg, Rydqvist and Sundaresan (2002) note that the Treasury departments of many countries, e.g. Switzerland, Mexico, Sweden, Finland, Germany, Norway, and Italy,[1] adjust the quantity in response to the bidding in their regular auctions for government debt. Further, Keloharju et al. (2005) comment that the Treasury of Finland announces which security will be offered for sale, but not the amount. Supply is determined after observing the bids, and it is influenced by an array of factors, among which are market conditions, the Treasury's opinion of the true market price, and the unwillingness to spoil the market by accepting very low bids.

The performance of variable supply auctions has attracted much theoretical interest, and the literature on this topic has been exciting and fast-evolving. Several recent contributions (e.g. Back and Zender 2001, Damianov 2005, McAdams 2007) study the effects of determining supply ex post on the equilibrium price in the uniform price auction. They consider a model of a perfectly divisible good in which the marginal cost of the seller is known to the bidders. Major conclusion of these studies is that, by varying supply ex post, the seller can substantially reduce or completely eliminate the low-price equilibria that exist in the fixed supply uniform price auction (see Wilson 1979, Back and Zender 1993). LiCalzi and Pavan (2005) further demonstrate that the seller can mitigate the low-price equilibria by committing to an increasing supply curve prior to the auction, and Kremer and Nyborg (2004*a*) find that underpricing can be eliminated with the pro rata allocation rule even when supply is fixed.

Complementary to this literature, this paper compares the uniform price and the discriminatory auction in an incomplete information setting in which the seller acts strategically after observing the bids. Our analysis provides a ranking of these two commonly used auction formats in terms of bids, revenue for the seller, and average trading volume.

We assume that a monopolist with constant marginal production cost and no capacity constraint offers multiple units of a good to two or more buyers. The buyers are risk-neutral and face uncertainty about the marginal cost of the seller. In the first stage of the game, bidders simultaneously submit their individual bids for one unit to the auctioneer. In the second stage, given the received bids and the production cost, the seller decides on

[1]See Heller and Lengwiler (2001), Umlauf (1993, pp. 316–317), Nyborg et al. (2002), Keloharju, Nyborg and Rydqvist (2005) and Rocholl (2004) for descriptions of the Treasury auction procedures in Switzerland, Mexico, Sweden, Finland, and Germany, respectively. In these countries, the Treasuries use a discriminatory auction and determine supply ex-post. Scalia (1997) and Bjonnes (2001) report that the Treasury auctions in Norway and Italy, respectively, are uniform price auctions with variable supply.

a supply quantity so as to maximize profit. In the discriminatory auction, the seller acts as a perfectly discriminating monopolist with respect to the received bids, while in the uniform price auction the seller charges all winning bidders the same price determined by the lowest winning bid.

We show that in every symmetric (mixed strategy) equilibrium buyers bid higher in the uniform price auction with a probability of one. In the two-bidder case this result holds for all rationalizable bids. As a result, the uniform pricing leads to a higher expected revenue for the seller. We also find under a convexity condition that the uniform price auction generates a higher trade volume on average.

At first sight these results may seem counterintuitive: standard monopoly theory tells us that a price discriminating monopolist sells a higher quantity and realizes higher profits. The major difference here, of course, is that bidders act strategically when submitting their bids. In a subgame perfect equilibrium, when bidders anticipate price discrimination, they will adjust their bids in order to counterbalance the discriminatory power of the seller. We will show that the commitment not to price discriminate in the uniform price auction will promote competition among bidders and eventually raise bid prices. As a consequence, the seller will be able to charge higher prices and sell a higher quantity.

To gain intuition into the way supply uncertainty affects bids in our model, we examine how a higher bid influences the chances of winning and the payment of a bidder in uniform price and discriminatory auctions. By submitting high bids, bidders can avoid (part of) the risk of not being served in the cases of high marginal cost, in which there will be a reduction in supply. That means, for both types of auctions, raising a bid raises the probability of winning. Increasing a bid, however, is less costly in the uniform price auction: while in the discriminatory auction winning bids are paid with a probability of one, in the uniform price auction all winners pay the lowest winning bid. So, in the uniform price auction, bidders with higher bids free ride on their lower bidding counterparts, since all bidders pay the same amount. This creates a tendency toward higher bidding in the uniform price auction.

In our setting the discriminatory auction is quite easy to analyze. Due to the lack of a capacity constraint, the seller optimally serves all bids above marginal cost, and there is essentially no competition among bidders. In the uniform price auction, in contrast, the commitment of the seller not to price discriminate may lead him to reject bids above marginal cost. This generates competition among bidders.

Our modeling approach is closely related to the paper of Lengwiler (1999). Similarly to Lengwiler, we assume that the seller has constant marginal cost of production and faces no capacity constraint. Further, in line with Lengwiler's model, we assume that marginal cost is private information of the seller. At the time of bidding buyers know only the distribution of the marginal cost. In his model, Lengwiler restricts bidders' choices to the announcement of bid quantities at two exogenously given prices – high and low. He proves that the uniform price and the discriminatory auctions have perfect equilibria. Since the characterization of equilibria in the setting of Lengwiler is rather difficult, both auctions could not be compared in terms of revenue for the seller or efficiency.

We reach here more definitive conclusions about revenue and efficiency for two reasons. First, we focus on prices and assume that each bidder desires a single item. This assumption helps us identify the tradeoff between probability of winning and payment. Second, we use a new methodological approach. We identify bounds on the set of rationalizable bids in the two-bidder case and the set of symmetric mixed strategy equilibria in the general case. This allows us to compare the two auctions without the need to calculate the equilibria explicitly.

According to McAdams (2007), the lack of a capacity constraint seems applicable to auctions of financial assets or procurement auctions where the cost of production is usually incurred after the auction. In this case the production capacity can often be adjusted to the demand. For those scenarios the assumption of constant marginal cost and unlimited capacity may be a reasonable approximation. Of course, the assumption does not describe well auctions in which there is a natural capacity constraint on the items available, such as auctions for landing slots at airports or electromagnetic spectrum.

We consider our work primarily as a theoretical contribution that reveals the effects of variable supply on equilibrium bids under uniform and discriminatory pricing. Recent empirical studies (e.g. Nyborg et al. 2002, Keloharju et al. 2005) put forward the idea that the data on bidding reflect bidders' optimal adjustments to supply uncertainty. Our model illustrates that supply uncertainty causes bids to rise more in the uniform price auction than in the discriminatory auction.

The results in the current paper are supported by a recent experimental study (see Damianov, Oechssler and Becker 2009). It compares bidding in the uniform and the discriminatory auction in a two-bidder setting featuring the type of supply uncertainty we consider here. In line with our theory, experimental subjects bid substantially higher in the uniform price treatment. The differences in seller revenue and average trade volume between the two treatments are significant and favor uniform pricing.

The rest of the paper is organized as follows. Section 2 presents the assumptions of the model and the theoretical framework. The analysis and the results can be found in Section 3. Section 4 contains a discussion of the results, and Section 5 concludes.

2 The model

A monopoly seller offers multiple units of an asset to $n \geq 2$ prospective buyers. The asset has a common value v to the buyers, and the seller has no information about that common value.[2] The seller will use a variable supply auction to sell the good. Each buyer $i \in \{1, 2, \ldots, n\}$ is risk neutral and submits a price bid for a single unit. The monopolist observes privately his constant marginal cost c, only the distribution of which is known to bidders. This distribution has support $[0, \overline{c}]$, where $\overline{c} \geq v$. The distribution function is denoted by $F(c)$ and its density function by $f(c)$. The latter is taken to be continuous, strictly positive in the interval $[0, \overline{c}]$ and differentiable in the interval $(0, \overline{c})$. Further, it is assumed log-concave, i.e.

$$\left(\ln F(c)\right)' = \frac{f(c)}{F(c)}$$

is a monotonically decreasing function.[3] We assume also that bidders are not able to pay infinitely large bid prices, that is, bids are restricted to an interval $[0, m]$, where $m > v$ is an arbitrarily large, but finite number.

In a variable supply auction, after receiving the bids, the seller decides on the supply quantity so as to maximize profit. We model this scenario as a two-stage game. We first introduce some general notation for the players' payoffs and provide a standard definition of equilibrium for an arbitrary variable supply auction game. Then we specify the payoffs separately for the uniform price and the discriminatory auction.

Pure strategies

Each bidder i submits a price bid x_i to the auctioneer, indicating the (highest) price he is willing to pay for a unit. The vector of submitted bids is denoted by $\mathbf{x}$ and the bid vector of all bidders except bidder i by $\mathbf{x}_{-i}$. Let us consider an arbitrary trade mechanism. Since the seller can condition the supply on the received bids, his strategy is a mapping from the set of bid vectors and possible values of the privately observed marginal cost c into supply quantity:

$$\phi\colon [0, m]^n \times [0, \overline{c}] \to \{0, 1, 2, \ldots, n\}.$$

Assume that, after observing bids $\mathbf{x}$ and marginal cost c, the seller supplies quantity q. We denote seller's profit by $r_{\mathrm{S}}(\mathbf{x}; q, c)$ and the payoff (or the net consumer surplus) of bidder i by $r_i(\mathbf{x}; q)$. If the seller supplies according to the strategy $\phi(\cdot)$, the expected

[2] A fully satisfactory approach would probably require that bidders have different privately observed signals of that common value. We will see that even in our simple model significant complexities arise especially in the analysis of the uniform price auction. In Section 4 we discuss that our results are robust to small asymmetries in the valuations of bidders, so this assumption is made primarily for notational clarity (for similar approach see e. g. Back and Zender 2001, Kremer and Nyborg 2004*a*, McAdams 2007).

[3] This property of the distribution implies a "monotone hazard rate", which is a standard assumption in auction theory. In single-unit first-price auctions it guarantees that bidders with higher valuations submit higher bids. It is satisfied by most of the common distributions: uniform, normal, logistic, chi-squared, exponential and Laplace. See Bagnoli and Bergstrom (2005) for a more complete list and for results allowing the identification of distributions with monotone hazard rates.

payoff of bidder i is

$$R_i(\mathbf{x};\phi) = \int_0^{\overline{c}} r_i\big(\mathbf{x};\phi(\mathbf{x},c)\big)\cdot f(c)\,\mathrm{d}c.$$

Mixed strategies

A mixed strategy σ_i of bidder i is a probability distribution over the set of pure strategies $[0,m]$. The set Σ of mixed strategies is the set of probability distributions defined on $([0,m],\mathscr{B})$, where $\mathscr{B}$ is the Borel σ-field on $[0,m]$. A mixed strategy profile of all bidders is denoted by $\boldsymbol{\sigma}$ and a mixed strategy profile of all fellow bidders of bidder i by $\boldsymbol{\sigma}_{-i}$. The expected payoff of bidder i when bidders play mixed strategies is defined as

$$\mathfrak{R}_i(\boldsymbol{\sigma};\phi) = \int R_i(\mathbf{x};\phi)\mathrm{d}\boldsymbol{\sigma}(\mathbf{x}).$$

Equilibrium

Our general analysis will focus on the subgame perfect equilibria of this two-stage market game.

Definition 1 (subgame perfect equilibrium)**.** *The mixed strategy profile $\boldsymbol{\sigma}^*$ and the supply function of the seller ϕ^* constitute a subgame perfect equilibrium (short: equilibrium), if the following conditions (SS) and (MS) hold.*

Second stage

For every vector of declared bids $\mathbf{x}$ and every value of the marginal costs c, the auctioneer sets the supply quantity so as to maximize profit:

$$\phi^*(\mathbf{x},c) \in \underset{q\in\{0,1,2,..,n\}}{\arg\max}\; r_{\mathrm{S}}(\mathbf{x},q,c). \tag{SS}$$

First stage

In the first stage of the game the strategy of every bidder i maximizes his expected payoff, given the strategies of the other bidders and the optimal supply function of the seller:

$$\mathfrak{R}_i(\sigma_i^*,\boldsymbol{\sigma}_{-i}^*;\phi^*) \geq \mathfrak{R}_i(\sigma_i,\boldsymbol{\sigma}_{-i}^*;\phi^*) \quad \forall\sigma_i\in\Sigma. \tag{MS}$$

Reduced game

We will further on consider only optimal behavior of the seller in the second stage of the trade mechanisms we analyze. From now on, we will, therefore, write

$$\mathfrak{R}_i(\sigma_i,\boldsymbol{\sigma}_{-i}) \quad \text{instead of} \quad \mathfrak{R}_i(\sigma_i,\boldsymbol{\sigma}_{-i};\phi^*),$$

always assuming that the seller supplies a profit maximizing quantity. We will similarly use $R_i(x_i, \mathbf{x}_{-i})$ instead of $R_i(x_i, \mathbf{x}_{-i}; \phi^*)$. Condition (MS) requires that bidders' strategies constitute a Nash equilibrium in the reduced game.

3 Analysis of the uniform price and the discriminatory auction

In both the uniform and the discriminatory auction the seller orders the bids in a descending order and serves them until the supply q is exhausted. Whereas in the uniform price auction all winning bidders pay a price equal to the lowest winning bid (called stopout price), in the discriminatory auction all winners are charged their own bid prices. Let us introduce some additional notation to describe the players' payoffs. Take an arbitrary bid vector $\mathbf{x}$. Order the bids in a *descending* order. For that purpose define the function

$$\varphi_{\mathbf{x}} : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\},$$

where $\varphi_{\mathbf{x}}(j) = k$ if bidder j submitted the k-th highest bid. If two or more bids are equal, then the function φ orders them arbitrarily. Further we define

$$\boldsymbol{\tau}(\mathbf{x}) = \big(\tau_1(\mathbf{x}), \tau_2(\mathbf{x}), \ldots, \tau_n(\mathbf{x})\big),$$

where $\tau_k(\mathbf{x})$ is the k-th highest bid if the bids are ordered in a descending order. The stopout price then is $\tau_q(\mathbf{x})$. The payoff of bidder i in the uniform price auction is

$$r_i^U(\mathbf{x}; q) = \begin{cases} v - \tau_q(\mathbf{x}) & \text{for } \varphi_{\mathbf{x}}(i) \le q, \\ 0 & \text{for } \varphi_{\mathbf{x}}(i) > q. \end{cases}$$

The payoff of bidder i in the discriminatory price auction is

$$r_i^D(\mathbf{x}; q) = \begin{cases} v - x_i & \text{for } \varphi_{\mathbf{x}}(i) \le q, \\ 0 & \text{for } \varphi_{\mathbf{x}}(i) > q. \end{cases}$$

The payoffs of the auctioneer in the uniform price and in the discriminatory auction are, respectively,

$$\begin{aligned} r_S^U(\mathbf{x}; q, c) &= q \cdot \big(\tau_q(\mathbf{x}) - c\big), \\ r_S^D(\mathbf{x}; q, c) &= \sum_{j=1}^{q} \big(\tau_j(\mathbf{x}) - c\big). \end{aligned}$$

In the following analysis we will assume that the auctioneer supplies the larger quantity when indifferent between two quantities. Of course, such an event occurs with a probability of zero because the marginal cost of the seller is assumed to be a continuous random variable. This assumption, therefore, is not essential for our results, and serves the purpose of completeness only.

Discriminatory auction (D)

Theorem 1. *In the discriminatory auction every bidder has a strictly dominant strategy* z_D*, given by the solution of the equation*

$$v - z = \frac{F(z)}{f(z)}. \tag{D}$$

This finding is graphically illustrated in Figure 1. The proof is quite straightforward due to the linear production cost assumption. The seller is acting as a perfectly discriminating monopolist with respect to the submitted bids. Since all winning bidders have to pay their bids, all bids which exceed (or are at least not lower than) the marginal cost c will be served.[4]

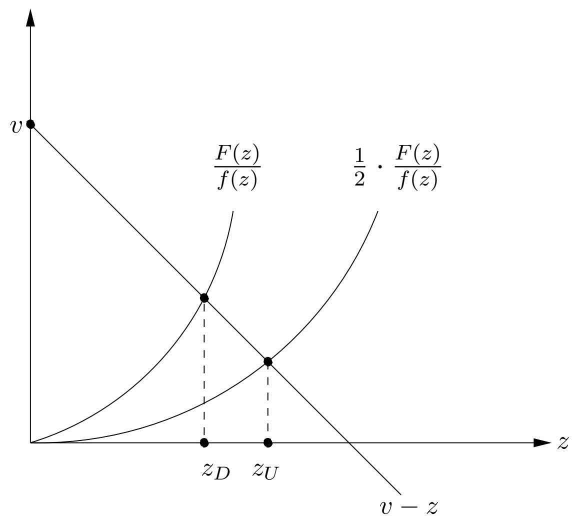

Figure 1: *z_D and z_U are the unique solutions of the equations (D) and (U) (see Theorem 1 and Theorem 2).*

The optimal supply quantity of the seller takes the form

$$\phi^*_{\mathrm{D}}(\mathbf{x}, c) = \max\{k : \tau_k(\mathbf{x}) \geq c\}.$$

[4]The auctioneer is indifferent between selling or not selling units to bidders who quoted a price equal to marginal cost. This detail is not important here as such events happen with zero probability because the distribution $F(c)$ is atomless.

For the payoff of bidder i we obtain

$$r_i^{\mathrm{D}}(\mathbf{x};\phi_D^*) = \begin{cases} v - x_i & \text{for } x_i \geq c, \\ 0 & \text{for } x_i < c. \end{cases}$$

The payoff of each bidder is independent of the other bids. The expected consumer surplus of bidder i is thus

$$R_i^{\mathrm{D}}(\mathbf{x}) = (v - x_i)F(x_i).$$

From the first order condition it follows that the maximizer z_D is the unique solution of equation (D). Existence and uniqueness[5] of z_D follow from the assumption that $F(c)$ is log-concave. In the bidding stage of the game, the bid z_D is a strongly dominant strategy for each player.[6] Since the payoff of each bidder is independent of how other buyers bid, there is indeed no competition in the discriminatory auction. Moreover, as all bidders have the same valuation for the good, they submit equal bids, and price discrimination does not materialize. As we will show, in the same scenario the uniform price auction promotes competition among bidders.

Uniform price auction (U)

The two bidder case

In this case $\mathbf{x} = (x_1, x_2)$, and with the notation we introduced, $\tau_1(\mathbf{x}) = \max\{x_1, x_2\}$; $\tau_2(\mathbf{x}) = \min\{x_1, x_2\}$. The payoff of the monopolist is

$$R_{\mathrm{S}}^{\mathrm{U}}(\mathbf{x}, q, c) = \begin{cases} 0 & \text{for } q = 0, \\ \tau_1(\mathbf{x}) - c & \text{for } q = 1, \\ 2(\tau_2(\mathbf{x}) - c) & \text{for } q = 2. \end{cases}$$

Second stage:

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[5]Consider the function $G(z) = v - z - \frac{F(z)}{f(z)}$. Observe that $G(0) = v > 0$ and $G(v) = -\frac{F(v)}{f(v)} < 0$ (F is log-concave). The continuity of $G(z)$ guarantees that the equation $G(z) = 0$ has a solution in the interval $(0, v)$ (by the Intermediate Value Theorem). The log-concavity of F requires that $\frac{F(z)}{f(z)}$ is a monotonically increasing function, therefore $G(z)$ is strictly monotonically decreasing. Thus the equation $G(z) = 0$ has a unique solution.

[6]Bidding z_D maximizes the buyer's expected surplus, so one might be tempted to think that the seller's expected revenue must be minimal for the discriminatory auction. As we will see, however, the auctions are not zero-sum games, because they lead to different average trading volumes, and thus different social surplus. So, a ranking cannot be provided on the basis of such an argument.

The optimal supply strategy of the auctioneer is given by

$$\phi_{\mathrm{U}}^{*}(\mathbf{x},c)=\begin{cases}0 & \text{for } c>\tau_1(\mathbf{x}),\\ 1 & \text{for } \tau_1(\mathbf{x})\geq c>2\cdot\tau_2(\mathbf{x})-\tau_1(\mathbf{x}),\\ 2 & \text{for } 2\cdot\tau_2(\mathbf{x})-\tau_1(\mathbf{x})\geq c.\end{cases}$$

First stage:
Now we can characterize the expected payoff of bidder i:
$R_i^{\mathrm{U}}(x_i,x_{-i})$

$$=\begin{cases}(v-x_i)\cdot\big(F(x_i)-F(2x_{-i}-x_i)\big)+(v-x_{-i})\cdot F(2x_{-i}-x_i) & \text{for } x_i\geq x_{-i},\\ (v-x_i)\cdot F(2x_i-x_{-i}) & \text{for } x_i<x_{-i}.\end{cases}\tag{3.1}$$

The next Theorem establishes several important properties of the expected payoff function. See Figures 1 and 2 for graphical illustrations of these properties.

Theorem 2. *The expected profit function has the following properties; $\partial_i^+R_i^{\mathrm{U}}(x_i,x_{-i})$ denotes the partial derivative from above with respect to x_i:*

(i) $R_i^{\mathrm{U}}(x_i,x_{-i})$ *is continuous in* (x_i,x_{-i}),

(ii) $R_i^{\mathrm{U}}(x_i,x_{-i})=0$ *for* $0\leq x_i\leq x_{-i}/2$,

(iii) $\partial_i^+R_i^{\mathrm{U}}(x_i,x_{-i})>0$ *for* $x_i=x_{-i}<v$,

(iv) $\partial_iR_i^{\mathrm{U}}(x_i,x_{-i})>0$ *for* $\frac{x_{-i}}{2}<x_i<\min\{x_{-i},z_U\}$, *where z_U is the unique solution of the equation*

$$v-z=\frac{1}{2}\cdot\frac{F(z)}{f(z)},\tag{U}$$

(v) There exists $\delta>0$ such that

$$\partial_iR_i^{\mathrm{U}}(x_i,x_{-i})>0\quad\text{for}\quad x_{-i}<x_i<\min\{x_{-i}+\delta,z_U\}.$$

Proof.
The statements (i) and (ii) follow directly from equation (3.1). To prove (i), observe that for $x_i=x_{-i}$ both lines in (3.1) are equal to $(v-x_i)\cdot F(x_i)$. The intuition behind (ii) is simple. If $0\leq x_i\leq x_{-i}/2$, it is not profitable for the seller to service bidder i for any realization of the marginal cost c, which means that with a probability of one bidder i is

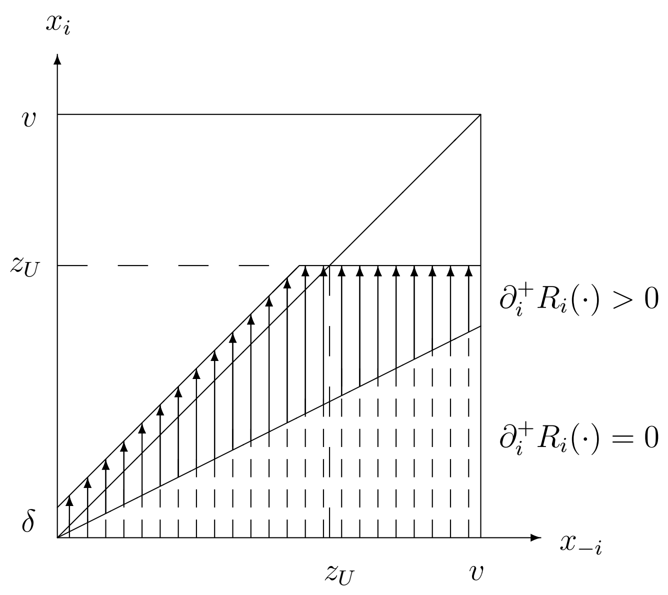

Figure 2: *In the dash-line area the payoff of bidder i is zero (see property (ii)). In the vector area the bidder's payoff increases in the direction of the arrows (see properties (iii), (iv) and (v) of Theorem 2).*

not served. To prove statement (iii), note that for $x_i = x_{-i}$ we obtain

$$\begin{aligned}\partial_i{}^+ R_i^{\mathrm{U}}(x_i, x_{-i}) = & \quad (v - x_i) \cdot \big(f(x_i) + f(2x_{-i} - x_i)\big) - F(x_i) \\ & \quad + F(2x_{-i} - x_i) - (v - x_{-i}) \cdot f(2x_{-i} - x_i) \\ = & \quad (v - x_i) \cdot f(x_i) > 0.\end{aligned}$$

Statement (iv) follows from the (in)equalities

$$\begin{aligned}\partial_i R_i^{\mathrm{U}}(x_i, x_{-i}) &= (v - x_i) \cdot f\,(2x_i - x_{-i}) \cdot 2 - F\,(2x_i - x_{-i}) \\ &= 2f\,(2x_i - x_{-i}) \left[(v - x_i) - \frac{F\,(2x_i - x_{-i})}{2f\,(2x_i - x_{-i})}\right] \\ &> 2f\,(2x_i - x_{-i}) \left[(v - x_i) - \frac{F\,(x_i)}{2f\,(x_i)}\right] \\ &> 2f\,(2x_i - x_{-i}) \left[(v - z_U) - \frac{F\,(z_U)}{2f\,(z_U)}\right] = 0.\end{aligned}$$

Notice that the last two inequalities apply because, as assumed, $\frac{F}{f}$ is a monotonically increasing function. A rigorous proof of property (v) can be found in Appendix A. Here we illustrate only the main idea. We exploit the already established properties (i) and (iii) and the fact that pre-images of open sets under continuous mappings are open to reach the conclusion that in an open neighborhood around the set

$$\{(x_i, x_{-i}) \mid x_i = x_{-i} < z_U\}$$

the partial derivative from above with respect to x_i is positive. The claim follows. □

As a consequence of Theorem 2 and equation (3.1) we obtain the statement:

Corollary 1. *The (pure strategy) best response correspondence x_i^* of each bidder i, has the following properties:*

$$x_{-i} \notin x_i^* (x_{-i}), \tag{3.2}$$

$$x_i^* (0) = \{z_D\}, \tag{3.3}$$

$$x_i < v \quad \text{for all} \quad x_i \in x_i^* (v). \tag{3.4}$$

Proof. (3.2) follows from (iii); (3.3) and (3.4) follow from (3.1). □

(3.2) implies that the uniform price auction has no symmetric subgame perfect equilibrium in pure strategies. (3.3) and (3.4) further imply that the best response correspondence is not continuous, which points to the generic difficulty for the existence of pure strategy equilibria. Indeed, if the best response were continuous, it should cross the 45° line, which does not happen here because of (3.2). In subsection 3 we calculate the best response for an example of uniformly distributed marginal cost. See Figure 6 for an illustration of the best response correspondences for that numerical example. The next Theorem provides an equilibrium existence result.

Theorem 3 (equilibrium existence)**.** *The uniform price auction has a mixed strategy equilibrium.*

Proof. The existence is guaranteed by Glicksberg's (1952) theorem, since the expected payoff function $R_i^{\mathrm{U}}(x_i, x_{-i})$ is continuous (see property (i)) and the support $[0, m]$ of the bids is a convex and compact set. □

Direct derivation of the mixed strategy equilibria in this setting is **quite challenging**, and we will, therefore, further examine rationalizable strategies in the two bidder case. This approach has at least three advantages. First, all equilibria belong to the set of rationalizable strategies, and the properties we will establish for rationalizable strategies are also valid for all equilibria. Second, our analysis will give a relatively sharp prediction for the size of the bids, which allows us to unambiguously rank the two auctions. Third, in two-player games the concepts of rationalizability and iterated strict dominance are equivalent (see Pearce 1984, Zimper 2005), and our result for the discriminatory auction, where we have a dominant strategy prediction, will be directly comparable with the results for the uniform price auction. Next we define rationalizable strategies.

Definition 2 (rationalizable strategies)**.** *Let* $\Sigma_i^0 := \Sigma$*, and for each* i *recursively define*

$$\Sigma_i^k = \Big\{ \sigma_i \in \Sigma_i^{k-1} : \exists \boldsymbol{\sigma}_{-i} \in \operatorname{conv} \Sigma_{-i}^{k-1} \quad \textit{such that} \\ \Re_i(\sigma_i, \boldsymbol{\sigma}_{-i}) \geq \Re_i(\sigma_i', \boldsymbol{\sigma}_{-i}) \qquad \textit{for all} \quad \sigma_i' \in \Sigma_i^{k-1} \Big\}.$$

$\operatorname{conv} \Sigma_{-i}^{k-1}$ *stands for the convex hull of the set* Σ_{-i}^{k-1}*, i.e. the smallest convex set that contains it. The set of rationalizable strategies*[7] *for player* i *is defined as*

$$\Sigma_i^{\text{rat}} = \bigcap_{k=0}^{\infty} \Sigma_i^k.$$

The rationalizable (or strategically sophisticated) strategy profiles are (mixed) strategy profiles which survive the serial deletion of strategies not belonging to the best responses of the players. Obviously, in a symmetric game the sets of rationalizable strategies for all players are equal. For notational brevity we will, therefore, omit the index i and simply use Σ^U for the set of rationalizable strategies in the uniform price auction. The set of rationalizable strategies and the set surviving the iterated deletion of strictly dominated strategies coincide in two-player games.[8] This property will be useful later on when we discuss the implications of Theorem 4.

Theorem 4 (rationalizable strategies)**.** *The set of rationalizable strategies of the uniform price auction contains only mixed strategies with support in the interval* $[z_U, v]$*:*

$$\sigma\big([z_U, v]\big) = 1 \quad \textit{for all} \quad \sigma \in \Sigma^U.$$

As noted earlier, the Theorem applies also for the sets of mixed strategies which survive the serial deletion of strongly dominated strategies. One can easily check[9] that $z_U > z_D$; therefore, it follows from Theorems 1 and 4 that the rationalizable bids in the uniform price auction are (almost surely) higher than those in the discriminatory auction. Before

[7]For brevity and ease of access, we stick to the definition and the notation of Fudenberg and Tirole (1991, p. 49, Definition 2.3). Although this definition does not introduce the notion of a belief system as the original definition does (see Bernheim 1984, pp. 1013–1014, Definitions 3.1–3.3), it is equivalent to Bernheim's (1984) definition. The only difference is that Fudenberg and Tirole (1991) consider only games with a finite strategy space (see also Pearce 1984), whereas Bernheim (1984), similarly to the model presented here, considers a more general strategy space, which is a compact subset of an Euclidean space.

[8]Easily accessible proofs can be found in Fudenberg and Tirole (1991, pp. 51–52) and Pearce (1984, pp. 1048–1049, Appendix B, Lemma 3). These proofs are conducted for games with finite strategy spaces, but the claim is also valid for the compact strategy sets of our model (for this argument consult Zimper 2005).

[9]This follows directly from the fact that z_D solves equation (D), z_U solves equation (U), and $F(c)$ is log-concave (see Figure 1).

we provide a proof of Theorem 4, let us explain why rational players bid higher in the uniform price auction. Consider the case in which bidder i has submitted a bid at least as high as his fellow bidder ($x_i \geq x_{-i}$), and let us compare the changes in his payoff resulting from an increase of his bid under the two pricing rules. In both auction formats the probability $F(x_i)$ of winning will clearly increase equally. While in the discriminatory auction the bidder has to pay his new bid with a probability of one, in the uniform price auction he pays on average less: he pays the bid price of his fellow bidder when both bidders are served. In this case, increasing his bid is more profitable (or at least less unprofitable) under the uniform pricing rule. Consider now the case $x_i < x_{-i}$. In this scenario bidder i is served with higher probability in the discriminatory than in the uniform price auction: $F(x_i) > F(2x_i - x_{-i})$. In the uniform price auction the bidder, therefore, will want to compensate for this lower probability by increasing his bid. This very characteristic of the uniform price payment rule creates incentives for higher bidding.

We now move on to provide the main idea of the proof of the Theorem 4. The rigorous but technical proof can be found in Appendix A.

Sketch of proof of Theorem 4 (rationalizable strategies).

It is intuitive that rational players do not bid higher than their valuation (see Part 1 of the proof in Appendix A). The more interesting part is to show that bidders do not bid below z_U. Property (v) of Theorem 2 reads:

There exists $\delta > 0$ such that

$$\partial_i R_i^{\mathrm{U}}(x_i, x_{-i}) > 0 \quad \text{for} \quad x_{-i} < x_i < \min\{x_{-i} + \delta, z_U\}.$$

This property is depicted in Figure 2, where we can see that in a small neighborhood above the 45° line the payoff of bidder i is increasing in his bid. We use now this property to state:

Bidders who play rationalizable strategies do not bid below z_U, i.e.

$$\sigma\big([0, z_U]\big) = 0 \quad \textit{for all} \quad \sigma \in \Sigma^U.$$

The proof of this statement is relegated to Appendix A (see Part 2 of the proof therein). The idea is to divide the interval $[0, z_U)$ into small intervals of length δ (in the sense of statement (v) of Theorem 2), where $z_U/\delta = N$ is an integer number. We denote the intervals

$$I_k = [(k-1) \cdot \delta, k \cdot \delta) \quad \text{for} \quad k = 1, 2, \ldots N; \quad I_0 = \emptyset,$$

as illustrated in Figure 3. By an iterative procedure we show that mixed strategies placing positive probability on $I_1, I_2, \ldots, I_N$ are not rationalizable. For that purpose we use the properties of bidders' payoff functions as stated in Theorem 2.

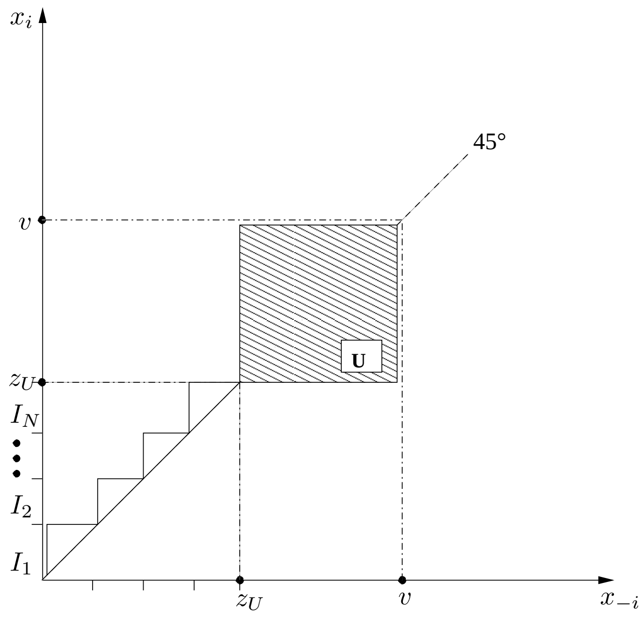

Figure 3: *The dark colored rectangle depicts the boundaries of the support of the rationalizable strategies in the uniform price auction. The triangles illustrate the serial elimination of mixed strategies placing positive probability on the intervals $I_1, I_2, \ldots, I_N$.*

The general case

In this subsection we discuss the case in which an arbitrary number of $n \geq 2$ bidders participate in the uniform price auction. We formally derive bidders' payoff function and show that it is continuous in the vector of declared bids (see Lemma 1).
This finding is used to verify that the uniform price auction has a symmetric mixed strategy equilibrium (Theorem 5). We further prove that in a symmetric equilibrium bids in the uniform price auction are with a probability of one higher than bids in the discriminatory auction (Theorem 6).

First, we derive bidders' payoff function. Let $\mathbf{x}$ be an arbitrary bid vector and $q, q' \in \{0, \ldots, n\}$. Define $\tau_0(\mathbf{x}) := v$. The seller weakly prefers to sell q instead of q' units if and only if

$$\big(\tau_q(\mathbf{x}) - c\big) \cdot q \geq \big(\tau_{q'}(\mathbf{x}) - c\big) \cdot q'.$$

Thus the seller will not supply *more* than q units if and only if

$$c \geq c_q^-(\mathbf{x}) := \begin{cases} \max_{q<q'\leq n} \frac{q\cdot\tau_q(\mathbf{x})-q'\cdot\tau_{q'}(\mathbf{x})}{q-q'} & \text{for } q<n, \\ 0 & \text{for } q=n. \end{cases}$$

He will not supply *less* than q units if and only if

$$c \leq c_q^+(\mathbf{x}) := \begin{cases} \min_{0\leq q'<q} \frac{q\cdot\tau_q(\mathbf{x})-q'\cdot\tau_{q'}(\mathbf{x})}{q-q'} & \text{for } q\geq 1, \\ \bar{c} & \text{for } q=0. \end{cases}$$

So, the seller optimally supplies the quantity q for $c \in [c_q^-, c_q^+]$. The set of winners is then $\{j \mid \varphi_{\mathbf{x}}(j) \leq q\}$, and all winners pay the stopout price $\tau_q(\mathbf{x})$. The expected payoff of an arbitrary bidder i is thus

$$R_i^{\mathrm{U}}(\mathbf{x}) = \sum_{q=0}^{n} \big(v - \tau_q(\mathbf{x})\big) \cdot P(q;\mathbf{x}) \cdot \mathbb{1}_{\{\varphi_{\mathbf{x}}(i)\leq q\}},$$

where

$$P(q;\mathbf{x}) := \text{Prob } \big(c_q^+(\mathbf{x}) > c > c_q^-(\mathbf{x})\big) = \max\big\{F\big(c_q^+(\mathbf{x})\big) - F\big(c_q^-(\mathbf{x})\big), 0\big\}$$

is the probability that exactly q units are sold. Now we can state:

Lemma 1 (continuity)**.** *$R_i^{\mathrm{U}}(\mathbf{x})$ is continuous in* $\mathbf{x}$.

See Appendix A for a proof.

Theorem 5 (existence)**.** *The uniform price auction has a symmetric mixed strategy equilibrium.*

The Theorem follows immediately from Becker and Damianov (2006). There it is shown that symmetric games with continuous payoffs and compact and convex strategy spaces possess symmetric mixed strategy equilibria. Now we can formulate our main result for the general case:

Theorem 6. *In every symmetric equilibrium $\boldsymbol{\sigma}_U^*$ of the uniform price auction, bids are almost always higher than in the discriminatory auction:*

$$\sigma_U^*\big((z_D, v]\big) = 1.$$

Sketch of the proof. Although the idea of the proof is simple, the proof itself is quite technical and lengthy. It is relegated to Appendix B. Here we provide only the basic intuition and sketch the most important argument. It is clear that in every symmetric mixed strategy equilibrium buyers do not bid higher than their valuation. The more interesting part is to show that in every symmetric equilibrium buyers bid almost surely higher than z_D. We denote by z_* the lower bound of the support of a symmetric mixed strategy equilibrium:

$$z_* = \max\{z \mid \sigma_U^*\big([z, v]\big) = 1\}.$$

The proof proceeds by contradiction. We assume that there exists a symmetric mixed strategy equilibrium for which $z_* \leq z_D$. We consider a deviation strategy of an arbitrary bidder i, which shifts the probability mass of an interval $Z_*^\varepsilon := [z_*, z_* + \varepsilon)$ to the point $z_* + \varepsilon$ and show that for a sufficiently small $\varepsilon > 0$ the deviation is profitable. Thus we reach a contradiction to the equilibrium assumption.

Revenue and average trade volume

Theorem 4 stated that the supports of the rationalizable strategy sets in the uniform price and the discriminatory auction are disjoint in the two-bidder case. In Theorem 6, we further argued that in the general case the supports of the symmetric mixed strategy eqilibrium sets in the two auctions are disjoint. Bidders submit higher bids in the uniform price auction with a probability of one. As a consequence of these Theorems, we obtain a ranking of the auction formats in terms of revenue for the auctioneer and efficiency.

Revenue

Recall that $r_{\mathrm{S}}(\mathbf{x}; q, c)$ denotes the seller's profit, where $\mathbf{x}$ is the vector of submitted bids, q is the supply quantity and c is the seller's marginal cost. If bidders play the mixed strategy profile $\boldsymbol{\sigma}(\mathbf{x})$, the seller's expected revenue is given by

$$\Re_{\mathrm{S}}\big(\boldsymbol{\sigma}; \phi(\mathbf{x}, c), c\big) = \int r_{\mathrm{S}}\big(\mathbf{x}; \phi(\mathbf{x}, c), c\big) \mathrm{d}\boldsymbol{\sigma}(\mathbf{x}).$$

Let us denote by $\boldsymbol{z_D}$ the bidders dominant strategy profile in the discriminatory auction. The next theorem provides a revenue comparison of the two auctions.

Theorem 7 (revenue)**.** *For any given value of the marginal cost c, the expected revenue in the uniform price auction is at least as high as that in the discriminatory auction:*

(a) for all rationalizable strategies in the two-bidder case

$$\Re_{\mathrm{S}}^U(\boldsymbol{\sigma}_U; c) \geq \Re_{\mathrm{S}}^D(\boldsymbol{z_D}; c) \quad \textit{for} \quad n = 2 \quad \textit{and all} \quad \boldsymbol{\sigma}_U \in (\Sigma^U)^n, \qquad (R_2)$$

(b) for all symmetric mixed strategy equilibria in the general case

$$\mathfrak{R}_{\mathrm{S}}^{U}(\boldsymbol{\sigma}_U^*; c) \geq \mathfrak{R}_{\mathrm{S}}^{D}(\boldsymbol{z_D}; c) \quad \textit{for} \quad n \geq 2. \tag{R_n}$$

Expected revenue in the uniform price auction is strictly higher when positive quantities of the good are traded.

Proof. The uniform price auction generates higher expected revenue because bids are higher. For $c < z_D$ positive quantities will be traded both in the uniform price and in the discriminatory auction. In this case we can prove (R_n) by using Theorems 6 and 1:

$$\mathfrak{R}_{\mathrm{S}}^{U}(\boldsymbol{\sigma}_U^*; c) > r_S^U(\boldsymbol{z_D}; c) = r_S^D(\boldsymbol{z_D}; c) = \mathfrak{R}_{\mathrm{S}}^{D}(\boldsymbol{z_D}; c).$$

The proof of (R_2) is analogous; apply Theorems 4 and 1. For $c > z_D$ there will be no trade in the discriminatory auction. If there are bids in the uniform price auction which exceed c, then at least one of them will be served, and the uniform price auction will again be strictly more profitable. If all bids in the uniform price auction are below c, then there will be no trade in the two auction formats. This is the only case in which the uniform price and the discriminatory auction will generate the same revenue. □

Average trade volume

The average trading quantity[10] resulting from the mixed strategy profile $\boldsymbol{\sigma}(\mathbf{x})$ is given as follows:

$$\mathfrak{Q}(\boldsymbol{\sigma}) = \int Q(\mathbf{x}) \mathrm{d}\boldsymbol{\sigma}(\mathbf{x}),$$

where

$$Q(\mathbf{x}) = \int_0^v \phi^*(\mathbf{x}, c) \cdot f(c) \mathrm{d}c.$$

Theorem 8 (average trade quantity)**.** *If the marginal cost distribution function is convex ($F'' \geq 0$), the average trade quantity in the uniform price auction is higher than in the discriminatory auction:*

(a) for all rationalizable strategies in the two-bidder case

$$\mathfrak{Q}^U(\boldsymbol{\sigma}_U) > \mathfrak{Q}^D(\boldsymbol{z_D}) \quad \textit{for} \quad n = 2 \quad \textit{and all} \quad \boldsymbol{\sigma}_U \in (\Sigma^U)^n, \tag{E_2}$$

[10]Average trading quantity can be taken also as an efficiency measure. Note that buyers are only served when $v \geq c$, since they submit bids not higher than v and the seller does not serve bids below c. This means that trade takes place only when desirable ex-post. The mechanism which induces a higher probability for sale, i.e. higher average turnover, can, therefore, be considered as the more efficient mechanism. We have to point out, however, that higher average turnover does not necessarily imply higher efficiency in Pareto sense or ex-ante higher sum of the surplus of market participants.

(b) for all symmetric mixed strategy equilibria in the general case

$$\mathfrak{Q}^U(\boldsymbol{\sigma}^*_U) > \mathfrak{Q}^D(\boldsymbol{z_D}) \quad \textit{for} \quad n \geq 2. \tag{E_n}$$

Proof. Recall that $P(q;\mathbf{x})$ denotes the probability with which the seller will sell quantity q if the vector of bids is $\mathbf{x}$. The average quantity sold in the uniform price auction can be expressed as a function of the ordered bids:

$$Q_U(\mathbf{x}) = Q_U(\boldsymbol{\tau}(\mathbf{x})) = \sum_{q=1}^{n} q \cdot P(q;\mathbf{x}) = \sum_{\{q|c_q^+ > c_q^-\}} q \cdot P(q;\mathbf{x}).$$

The last equality implies that we need to sum only over the elements $\tau_q(\mathbf{x})$ for which $c_q^+ > c_q^-$, as otherwise $P(q;\mathbf{x}) = 0$. We write these quantities in an ascending order $l_1, l_2, \ldots, l_h$ and obtain

$$\tau_{l_1}(\mathbf{x}) > \tau_{l_2}(\mathbf{x}) > \cdots > \tau_{l_h}(\mathbf{x}).$$

For the sake of brevity, we will further write τ_{l_k} instead of $\tau_{l_k}(\mathbf{x})$. We will show that

$$Q_U(\tau_{l_1}, \tau_{l_2}, ..., \tau_{l_h}) \geq Q_U(\tau_{l_2}, \tau_{l_2}, ..., \tau_{l_h}). \tag{3.5}$$

We observe that $c_{l_1}^- = c_{l_2}^+$ and recall that $c_{l_1}^-$ is a solution of the equation

$$l_1 \cdot (\tau_{l_1} - c_{l_1}^-) = l_2 \cdot (\tau_{l_2} - c_{l_1}^-), \tag{3.6}$$

which means that the two dark rectangles in Figure 4 cover equal areas. Equation (3.6) is equivalent to

$$\frac{\tau_{l_1} - c_{l_1}^-}{\tau_{l_2} - c_{l_1}^-} = \frac{l_2}{l_1}.$$

From the convexity of F, it follows that

$$\frac{F(\tau_{l_1}) - F(c_{l_1}^-)}{F(\tau_{l_2}) - F(c_{l_1}^-)} \geq \frac{\tau_{l_1} - c_{l_1}^-}{\tau_{l_2} - c_{l_1}^-},$$

and we obtain

$$\frac{F(\tau_{l_1}) - F(c_{l_1}^-)}{F(\tau_{l_2}) - F(c_{l_1}^-)} \geq \frac{l_2}{l_1}. \tag{3.7}$$

The identities

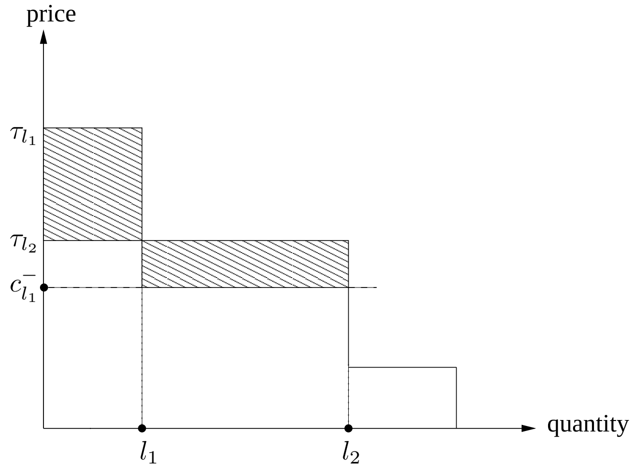

Figure 4: *Announced demand curve in the uniform price auction. The two pattern rectangles cover equal areas.*

$$Q_U(\tau_{l_1}, \tau_{l_2}, ...\tau_{l_h}) - Q_U(\tau_{l_2}, \tau_{l_2}, ...\tau_{l_h})$$

$$\begin{aligned} &= l_1\big[F(\tau_{l_1}) - F(\tau_{l_2})\big] - (l_2 - l_1)\big[F(\tau_{l_2}) - F(c_{l_1}^-)\big] \\ &= l_1\big[F(\tau_{l_1}) - F(c_{l_1}^-)\big] - l_2\big[F(\tau_{l_2}) - F(c_{l_1}^-)\big] \geq 0 \end{aligned}$$

prove (3.5). The above argument can be applied iteratively $(h-1)$ times to verify the inequality

$$Q_U(\tau_{l_1}, \tau_{l_2}, ...\tau_{l_h}) \geq Q_U(\tau_{l_h}, \tau_{l_h}, ...\tau_{l_h}).$$

Now one can easily prove (E_n) by applying Theorem 6:

$$\begin{aligned} \mathfrak{Q}_U(\boldsymbol{\sigma}_U^*) = \int Q_U(\mathbf{x})\mathrm{d}\boldsymbol{\sigma}_U^*(\mathbf{x}) &\geq \int Q_U(\tau_h, \tau_h, ..., \tau_h)\mathrm{d}\boldsymbol{\sigma}_U^*(\mathbf{x}) \\ &> \int Q_U(z_D, z_D, ..., z_D)\mathrm{d}\boldsymbol{\sigma}_U^*(\mathbf{x}) = n \cdot F(z_D) \\ &= \mathfrak{Q}_D(\boldsymbol{z_D}). \end{aligned}$$

The proof of (E_2) is analogous (apply Theorem 4). □

A numerical example

We consider the following two bidder example: $v = 1$ *and the marginal cost is uniformly distributed:* $f(c) = 1$ *for* $c \in [0, 1]$.

In the discriminatory auction, the dominant strategy for each bidder is $z_D = 1/2$ (Theorem 1). The support of the rationalizable strategies in the uniform price auction is contained

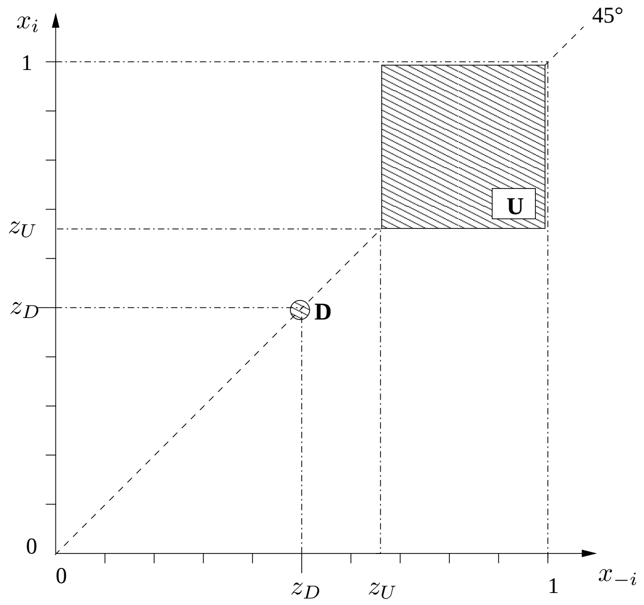

Figure 5: *Numerical example: $v = 1$, $n = 2$ and $f(c) = 1$ for $c \in [0, 1]$. The supports of the rationalizable strategy sets in the two auction forms (the pattern areas) are disjoint. Bids in the uniform price auction are higher with a probability of one.*

in the interval $[z_U, v] = [2/3, 1]$ (Theorem 2) as illustrated in Figure 5. Below we will derive the best responses of bidders and establish that the uniform price auction has two asymmetric pure strategy equilibria: $(\frac{9}{13}, \frac{10}{13})$ and $(\frac{10}{13}, \frac{9}{13})$.

Let us first gain some intuition for the higher equilibrium bids in the uniform price auction. In particular, let us clarify why the pair (z_D, z_D) is not an equilibrium in this auction. For that purpose, we explore how the payoff of a bidder (say bidder i) is affected by a marginal increase of his bid to $z_D + \varepsilon$. In the cases (1) $c > z_D + \varepsilon$ and (2) $c < z_D - \varepsilon$ the payoff is the same, independent of whether bidder i bids z_D or $(z_D + \varepsilon)$. In case (1) bidder i is not served, whereas in case (2) bidder i is served and pays z_D. A difference in the payoff is observed when $z_D - \varepsilon \leq c \leq z_D + \varepsilon$. In particular, by increasing his bid, bidder i *loses* ε when $z_D - \varepsilon \leq c \leq z_D$ (he is served but pays $z_D + \varepsilon$ instead of z_D), and *gains* $1 - (z_D + \varepsilon)$ when $z_D < c \leq z_D + \varepsilon$ (he is served with the bid $z_D + \varepsilon$ but not with the bid z_D). The probability for the events $z_D - \varepsilon \leq c \leq z_D$ and $z_D < c \leq z_D + \varepsilon$ is the same and equals ε, and for a small enough ε the gain of $1 - (z_D + \varepsilon)$ exceeds the loss of ε. The deviation is profitable. Observe that this analysis applies also to general distributions of c. For a small enough ε the probabilities of the events $z_D - \varepsilon \leq c \leq z_D$ and $z_D < c \leq z_D + \varepsilon$ are about the same and very close to $f(z_D)\varepsilon$. Additional insights can be gained by comparing *marginal gain* and *marginal cost* of increasing the bid of bidder i above z_D in the uniform and the discriminatory auction. The expected payoff of bidder

i in the discriminatory auction is

$$R_i^D(x_i) = (v - x_i)F(x_i) = vF(x_i) - x_iF(x_i).$$

The first term, v times the probability of winning, can be interpreted as the gain (or revenue) of bidding x_i, and the second term, payment times the probability of winning, can be interpreted as the cost of bidding x_i. Marginal gain and marginal cost of bidding x_i in the discriminatory and the uniform price auction are, respectively,

$$\begin{aligned} MG^D(x_i) &= \frac{d}{dx_i}\big(vF(x_i)\big) = vf(x_i), \\ MC^D(x_i) &= \frac{d}{dx_i}\big(x_iF(x_i)\big) = x_if(x_i) + F(x_i). \end{aligned}$$

The dominant strategy equilibrium bid z_D solves the equation $MG^D(x_i) = MC^D(x_i)$ (see Theorem 1). Consider now the uniform price auction. If bidder i bids $x_i \geq z_D$ and the other bidder bids z_D, the expected payoff for bidder i in the uniform price auction is

$$\begin{aligned} R_i^U(x_i, z_D) &= vF(x_i) - \big[z_DF(2z_D - x_i) + x_i\big(F(x_i) - F(2z_D - x_i)\big)\big] \\ &= vF(x_i) - \big[x_iF(x_i) + (z_D - x_i)F(2z_D - x_i)\big] \end{aligned}$$

The first term, v times the probability of winning, is the gain of bidding x_i, and the term in the squared brackets is the expected payment (or cost of bidding x_i). Marginal gain in the uniform price auction, $MG^U(x_i)$, is the same as in the discriminatory auction. For the marginal cost when $x_i = z_D$ we obtain

$$\begin{aligned} MC^U(x_i, z_D)\Big|_{x_i=z_D} &= \frac{\partial}{\partial x_i}\big[x_i(F(x_i) + (z_D - x_i)F(2z_D - x_i)\big]\Big|_{x_i=z_D} \\ &= MC^D(z_D) - F(z_D) \\ &< MC^D(z_D) = MG^D(z_D) \\ &= MG^U(z_D). \end{aligned}$$

The marginal cost of increasing x_i above z_D is smaller in the uniform price auction than in the discriminatory auction, and hence smaller than the marginal gain (which is the same in both auctions). It follows that increasing x_i above z_D is profitable.

The average trade quantities of the discriminatory auction and the uniform price auction are

$$\mathfrak{Q}_D = \int_0^{\frac{1}{2}} 2\mathrm{d}c = 1, \quad \mathfrak{Q}_U > \frac{4}{3}.$$

The seller's revenue is clearly higher in the uniform price auction. The payoff of bidder i in the uniform price auction is

$$R_i^{\mathrm{U}}(x_i, x_{-i}) = \begin{cases} (1-x_i)x_i & \text{for } x_i > 2x_{-i}, \\ (1-x_i)\big(2x_i - 2x_{-i}\big) + (1-x_{-i})(2x_{-i} - x_i) & \text{for } 2x_{-i} \geq x_i \geq x_{-i}, \\ (1-x_i)(2x_i - x_{-i}) & \text{for } \frac{x_{-i}}{2} \leq x_i < x_{-i}, \\ 0 & \text{for } x_i < \frac{x_{-i}}{2}. \end{cases}$$

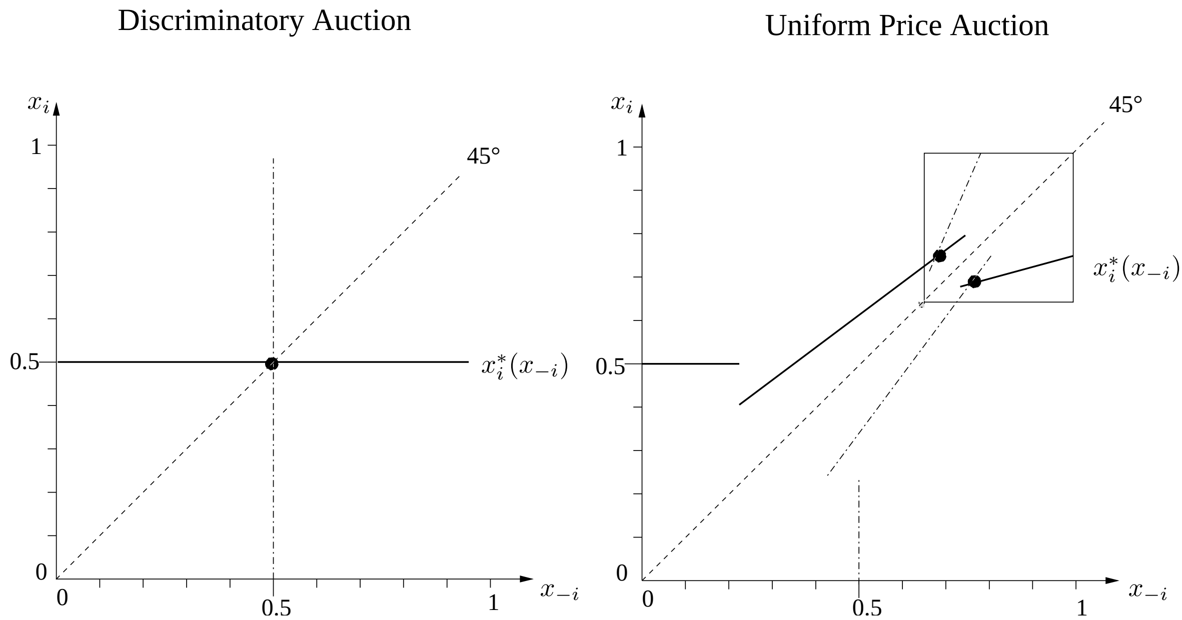

Figure 6: *Best responses and pure strategy equilibria (the thick dots) in the uniform price and the discriminatory auctions. The support of the rationalizable strategies of the uniform price auction lies within the square as has been proven in Theorem 4.*

For the (pure strategy) best response correspondence of bidder i in the uniform price auction we obtain

$$x_i^*(x_{-i}) = \begin{cases} \frac{1}{2} & \text{for } x_{-i} < \frac{3-\sqrt{2}}{7}, \\ \left\{\frac{16-3\sqrt{2}}{28}, \frac{1}{2}\right\} & \text{for } x_{-i} = \frac{3-\sqrt{2}}{7}, \\ \frac{3x_{-i}+1}{4} & \text{for } x_{-i} = \left(\frac{3-\sqrt{2}}{7}, \frac{3}{4}\right), \\ \left\{\frac{11}{16}, \frac{13}{16}\right\} & \text{for } x_{-i} = \frac{3}{4}, \\ \frac{x_{-i}+2}{4} & \text{for } x_{-i} \in \left(\frac{3}{4}, 1\right]. \end{cases}$$

The graph of the best responses of the two players is given in Figure 6. As noted, the uniform price auction has two asymmetric subgame perfect equilibria in pure strategies:

$$(x^*_{i,U}, x^*_{-i,U}) \;\; = \left(\tfrac{10}{13}, \tfrac{9}{13}\right), \;\; i = 1, 2.$$

For the average trade quantity in those equilibria we obtain

$$\mathfrak{Q}_U = \frac{18}{13} > \frac{4}{3}.$$

Seller's revenue is clearly higher in the uniform price auction.

4 Discussion

This paper presents a highly stylized model of an auction in which the seller controls supply ex post. Our main result is that the seller's ability to vary supply after observing the bids creates a greater tendency for bids to rise in the uniform price auction than in the discriminatory auction. Our analysis captures the interplay of two factors - bidders' uncertainty about marginal cost, and seller's profit maximizing decision. In the absence of uncertainty, bidders would submit bids equal to marginal cost both in the uniform price and in the discriminatory auction. Thus, the positive profits of the seller accrue from the information asymmetry between the bidders and the seller. Without a capacity constraint on the units available, in the discriminatory auction bidders do not compete with one another because the seller optimally serves all bids above marginal cost. In the uniform price auction, in contrast, the seller may find it optimal to reject some bids above marginal cost. Thus, the inability to price discriminate may lead the seller to reduce supply, and this feature generates competition.[11] In other words, in the uniform price auction bids are shifted farther toward the bidders' true valuation of the good, and the seller is able to exploit the uncertainty of the bidders more fully. This finding might be an important guide for empirical studies.

In the next subsection we explore the limitations of our model and discuss to what extent our results are applicable to more general settings. Then we compare our theoretical insights with the recent empirical findings on Treasury auctions. Finally, we derive the optimal mechanism in our theoretical setting.

Critical assumptions and extensions

Three assumptions play a critical role in our analysis. We will discuss them to assess the robustness of our results to alternative model specifications.

[11] We would like to thank an anonymous referee for suggesting this short and very intuitive explanation of our results.

1) *Constant marginal cost/no capacity constraint*

To understand the role of the constant marginal cost assumption, consider first the case of a publicly known binding capacity constraint (i.e. the number of bidders exceeds the number of units available). This specification is a special case of a weakly increasing marginal cost schedule in which the cost of every unit above the capacity constraint rises to infinity. With a binding capacity constraint, the equilibrium price will be v and the seller will sell all available units in both auctions. Indeed, assume this is not the case towards contradiction. If the buyers bid different amounts, the lowest bidder will not be served (if there are several lowest bidders and ties are resolved randomly, then each of them will not be served with a positive probability). In this case it will be profitable for this bidder to raise his bid marginally because his probability of winning an item will rise discontinuously. If all bids are equal and below v (ties are resolved randomly), then even when seller's marginal cost is below these bids, there is a positive probability for each bidder not to be served because of the capacity constraint. A marginal increase in a bid will lead to that bidder being served with a probability of one – a profitable deviation. Thus, all bidders submit v, and the two auctions generate the same revenue. So, a binding capacity constraint can be considered as a trivial limiting case of our model in which the uniform price auction is no longer superior, yet does not perform worse than the discriminatory auction either. Now consider the general case in which the seller faces an arbitrary increasing marginal cost function. In the discriminatory auction, the seller may want to service only a subset of the bidders. In this case the bidders' decisions in the discriminatory auction would no longer be independent of the other bidders' behavior, and there would be competition among the bidders. How the equilibria look like would depend on the way the uncertainty is modeled. The basic intuition seems to carry over: raising a bid raises the winning chances in the two auctions in a similar way, but is more costly in the discriminatory auction. This informal argument favors uniform pricing. Providing equilibrium existence results and obtaining closed form solutions for equilibrium bids is quite challenging and contains a host of new problems. As pointed out by Lengwiler (1999), this makes the assumption of elastic supply analytically attractive.

2) *Single-unit demand*

It is well known that the uniform price auction with a multi-unit demand and a fixed supply has low-price equilibria (see Wilson 1979, Back and Zender 1993). By submitting steep demand schedules, bidders make the acquisition of additional units by their competitors quite costly. That is, a small increase in the quantity assigned to a bidder can only be obtained at the cost of a substantial increase in the stop-out price. This feature of the uniform price auction inhibits competition and results in the existence of low-price equilibria. The questions whether these equilibria are sensitive to the model specification and whether the seller can alleviate the low price equilibrium problem by changing the allocation rules have recently received much attention in the theoretical literature.

Back and Zender (2001), Damianov (2005), and McAdams (2007) consider complete information settings in which the seller can vary supply after receiving the bids, and explore the effect of this assumption on the equilibrium bids. LiCalzi and Pavan (2005) demonstrate that the seller can raise his revenues by committing to an increasing supply schedule prior to the auction.

Kremer and Nyborg (2004*a*) show that the seller can eliminate the low price equilibria by using the pro-rata rationing rule, and Kremer and Nyborg (2004*b*) further demonstrate that if prices and quantities are discrete, then underpricing can be made arbitrarily small by adjusting the price tick size and the quantity multiple. The introduction of discrete prices and quantities and the use of the pro-rata rationing rule have an effect that is related to the effects arising in our model. The main purpose of the pro rata rationing rule or the discreteness of prices and quantities is to guarantee that a marginal increase in a player's bid will be sufficiently rewarded by a higher quantity allocated to that player. By making high bids less costly, these allocation rules promote competition. While in our model there is no such reward in the form of higher quantities, there is a reward in the form of a higher probability of winning an item. Increasing the probability of winning an item is less costly in the uniform price auction, as bidders sometimes pay a price that is lower than their bids.

Our revenue-ranking result is based on the assumption that bidders demand and can acquire only one item, and it is natural to wonder whether this result is robust to alternative formulations of the variable supply auction model. A question of particular interest is whether the revenue-ranking result will pertain if the bidders have multi-unit demand, or whether in that case the uniform price auction will be susceptible to the low-price equilibria known from the fixed supply models. If a bidder values all items received equally, then the analysis of the discriminatory auction remains unaltered. Due to the unlimited capacity and the lack of competition among bidders, all bidders will bid the same amount for all items. The analysis of the uniform price auction, however, is significantly more involved, and the assumption of a multi-unit demand adds another layer of complexity to an already intricate problem. When deciding how to bid on multiple items, the bidders face uncertainty regarding the bids submitted by the other bidders on the one hand, and uncertainty regarding the supply decision of the auctioneer on the other hand. In such a setting the derivation of optimal bids is particularly challenging. The analysis of a simple example of an auction with one bidder demanding two units reveals that it is optimal for the bidder to submit the same price for the two units. Allowing for the existence of bids submitted by other bidders, however, adds further complexity to the formulation of the payoff function and to the equilibrium analysis of the uniform price auction. The derivation of equilibria in variable supply auctions with multi-unit demand and supply uncertainty remains an important and challenging issue for future research.

3) *Equal valuations*

A key element of the comparisons is the assumption that all bidders share the same valuation. This assumption leads to low revenue in the discriminatory auction. Intuitively, the purpose of holding a discriminatory auction is discrimination among bidders. Price discrimination does not materialize here because all bidders bid the same amount. This result is robust to small differences in the information the bidders may have about the common value of the good. As long as the bidders' valuations are close, the bids will also be close and the discriminatory auction will be inferior from the seller's viewpoint. A more significant disparity in the valuations will result in more dispersed bids and favor the discriminatory auction.

Theory and empirical evidence on Treasury auctions

We draw two comparisons between our theoretical model and the empirical research on Treasury auctions. The first one concerns our assumptions, and the second one our results.

Let us focus first on seller's behavior. In the conclusion of their paper, Keloharju et al. (2005) search for an explanation of why the market power theory of uniform auctions (predicting existence of low price equilibria) has been rejected empirically. They attribute this empirical result to the strategic behavior of the seller, who participates actively in the price setting process rather than remaining passive as assumed in most of the theory. One particular explanation is that the Treasury may have an outside option to borrow from different sources or use different mechanisms if it is not happy with the bids. The authors report that conversations with Treasury officials reveal that their actual choices are influenced by (1) long-term revenue target, (2) market conditions, (3) the Treasury's opinion about the true market price, and (4) unwillingness to spoil the market by accepting very low bids (p. 1898). We believe this evidence gives some additional appeal to Lengwiler's, and to our assumption of privately observed marginal cost and the consequent supply uncertainty associated with this assumption.

Second, let us discuss the empirical evidence on bidding. Keloharju et al. (2005) compare underpricing of the uniform price Treasury auctions in Finland with the discriminatory Treasury auctions in Sweden. The find by and large that underpricing, defined as the difference between the secondary market price and the quantity weighted average price in the auction, is consistently lower in the uniform price auctions. For details see table X on p. 1896 and table XI on p. 1897. A similar result is also reported by Goldreich (2007), who analyzed US Treasury data on uniform price and discriminatory auctions. This result appears to conform with the predictions of our model. Yet, in contrast to our results, Keloharju et al. (2005) find a higher volatility of bids in the discriminatory auction. The underlying reason for this discrepancy may lie in our assumption of equal valuations: the uniform price auction is probably able to smooth out differences in the valuations, while in the discriminatory auction they become fully visible. It may also be due to other institutional features not captured by our model. We hope that future work will be able to clarify the reasons for this observation.

Discriminatory, uniform and optimal auctions

Our discussion so far focused on the two standard pricing rules – uniform and discriminatory. They are most often used in practice. Yet, in our setting they are not optimal from the aspect of efficiency, nor are they most profitable for the seller. Is there a mechanism that will allow the seller to extract the entire surplus from buyers?

Consider for example the following trade mechanism, which is a modification of the uniform price auction. As long as all bids are the same, and weakly exceed the seller's marginal cost, the seller serves all bidders (when bids are the same, but lower than the seller's marginal cost, none of them is served). If not all bids are the same, the seller does not serve the lowest bidder regardless of his bid (if more than one bidder define the lowest bid, the seller rations all of them) but otherwise behaves as in a standard uniform price auction.

It is easy to see that, with this modification, the asymmetric equilibria of the uniform price auction do not materialize. The lowest bidder will have an incentive to increase his bid so as to match the bid of other bidders in his attempt to avoid not being served. Additionally, if all bidders bid the same price below v, each bidder has the incentive to marginally increase his bid. This is inherent to the uniform price auction as we previously discussed. With these arguments it is clear now that the only equilibrium possible involves all buyers bidding v. Indeed, if all bidders behave that way no profitable deviation exists. This mechanism is revenue maximizing for the seller and efficient.[12]

Yet, compared to the two standard auctions, this mechanism requires a higher level of commitment on the seller's part: the seller does not serve the lowest bidder even when this is profitable. This type of commitment forces buyers to increase their bids up to v. To broaden this perspective, we can argue that, compared to the discriminatory auction, the uniform price auction is moving toward the optimal mechanism (not all the way though) by providing the seller with an incentive to ration the lowest bidder. The uniform pricing

[12]We would like to thank an anonymous referee for the suggestion to look for an optimal mechanism of this type in our model.

rule infringes on the seller's profit by making the object cheaper to all other bidders who submitted higher bids. As a result, the seller rejects some bids that exceed marginal cost. This is clearly not the case in the discriminatory auction where the seller has no commitment to serve bidders at a price below their bids, and the seller's revenue is lower in the discriminatory auction.

5 Conclusion

The standard pricing techniques, the uniform pricing rule, and the price discrimination rule are widely used by monopolists for the simultaneous sale of multiple units. When a monopolist lacks information about demand, these pricing techniques often take the form of an auction, in which the seller first collects bids from prospective customers and then decides on a supply quantity to maximize profit. These auction forms, called variable supply multi-unit auctions, are used on various markets ranging from Treasury bills and IPOs to emission permits and electricity. They differ from the fixed supply multi-unit auctions in the sense that the seller participates in the price-setting process, as he controls the supply after the bidding. We modeled this scenario as a two-stage game and compared these variable supply pricing mechanisms. In our setting the seller has privately observed constant marginal cost of production and faces no capacity constraint.

We found that, due to the supply uncertainty, in a symmetric equilibrium the bidders bid higher in the uniform price auction than in the discriminatory auction. This finding further implies that the uniform price auction is more profitable for the seller and leads to higher average trade volume.

The following intuition helps explain our results. In the discriminatory auction, due to the lack of a capacity constraint, the seller optimally serves all bids above marginal cost. Hence, the winning probability of each bidder is not affected by the bids of his fellow bidders. As the bidders share the same valuation, they submit equal bids. Thus, as in Lengwiler (1999), the right of the seller to discriminate among bidders and charge different prices has no bite. Bidders in the discriminatory auction do not compete at all; in the reduced form of the game a bidder's expected payoff is independent of how other bidders bid. In the uniform price auction, on the other hand, the probability of winning as well as the final price depend on all bids. Submitting higher bids in this auction format proves to be profitable as it raises the probability of winning, but not necessarily the price a bidder has to pay. This simple observation is used to demonstrate that the uniform price auction induces a more competitive environment and leads to higher equilibrium bids for any number of bidders (see Theorems 4 and 6). Our results are derived without having to compute the equilibria precisely. Instead, we exploited the properties of the bidders' payoff functions and the equilibrium and rationalizability concepts.

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A Appendix

Proof of property (v) of Theorem 2: *There exists $\delta > 0$ such that*

$$\partial_i R_i^{\mathrm{U}}(x_i, x_{-i}) > 0 \quad \text{for} \quad x_{-i} < x_i < \min\{x_{-i} + \delta, z_U\}.$$

On the set $K := \left\{(y_i, y_{-i}) \,\middle|\, 0 \le y_{-i} \le z_U, 0 \le y_i \le \frac{v - z_U}{2}\right\}$ we define a function $g\colon K \to \mathbb{R}$ by

$$g(y_i, y_{-i}) = \partial_i R_i^{\mathrm{U}}(y_i + y_{-i}, y_{-i})$$

where for $y_i = 0$ we mean the derivative from above. Notice that we simply expressed the partial derivative as a function of $y_{-i} = x_{-i}$ and the difference $y_i = x_i - x_{-i}$. The function g is continuous with $g(0, y_{-i}) > 0$ for every $y_{-i} \in [0; z_U]$, so the set $H := g^{-1}\big((0; \infty)\big)$ of points where the partial derivative is strictly positive is open[13] in K with $\{0\} \times [0, z_U] \subseteq H$. Therefore, as $[0, z_U]$ is compact, there exists[14] a neighborhood $[0, \delta]$, $\delta > 0$, of 0 in $[0, \frac{v - z_U}{2}]$ with $[0; \delta] \times [0, z_U] \subseteq H$.

Proof of Theorem 4:
Part 1: *Bidders do not bid higher than their valuation if they play rationalizable strategies:*

$$\sigma\big((v, m]\big) = 0 \quad \textit{for all} \quad \sigma \in \Sigma^U.$$

For each $\sigma \in \Sigma$, define $\hat{\sigma} \in \Sigma$ by

$$\hat{\sigma}(B) = \sigma\big(B \cap [0; v]\big) + \sigma\big((v; m]\big) \cdot \mathbb{1}_{v \in B} \quad \text{for } B \in \mathscr{B},$$

which means, a bidder with strategy $\hat{\sigma}$ bids v whenever a bidder with strategy σ would submit a bid from the interval $(v; m]$. We first notice that $\hat{\sigma}$ always weakly dominates σ, as bids above v lead to a strictly negative outcome when served. So a strategy σ_i of player i with $\sigma_i\big((v; m]\big) > 0$ will never be a best response to a strategy σ_{-i} of player $-i$, if player i has to pay more than v with strictly positive probability when the strategy combination (σ_i, σ_{-i}) is played.

Using this property, we will now show by induction that, with the notation of Definition 2, for $k = 1, 2, \ldots$

$$\sigma_i \notin \Sigma_i^{\mathrm{U},k}, \quad i = 1, 2, \quad \text{if } \sigma_i\Big(\big(\max\{v, 2^{-k} m\}; m\big]\Big) > 0. \tag{A.1}$$

We start with $k = 1$. As the other bid is never greater than m, bids from the inter-

[13]Here we use the fact that pre-images of open sets under continuous mappings are open, see e. g. Königsberger (2002), p. 16.
[14]This follows from the so called "tube lemma", see e. g. Königsberger (2002), p. 32.

val $\big(\max\{v, \frac{m}{2}\}; m\big]$ are served when the marginal cost of the seller is below v, which will happen with a strictly positive probability. So, by the introductory remark, if $\sigma_i\Big(\big(\max\{v, \frac{m}{2}\}; m\big]\Big) > 0$, $\hat{\sigma}_i$ will be strictly better than σ_i, regardless of what $-i$ does.

Now assume that equation (A.1) holds for $k-1$. Bids above $\max\{v, 2^{-k}m\}$ are served when the other bidder does not submit a bid above $\max\{v, 2^{-(k-1)}m\}$ and the cost is below v, which by induction happens with strictly positive probability if the other bidder plays a strategy from $\Sigma_{-i}^{\mathrm{U},k-1}$. So, for each strategy σ_i with $\sigma_i\Big(\big(\max\{v, 2^{-k}m\}; m\big]\Big) > 0$ the strategy $\widehat{\sigma_i}$ will be a strictly better response to any element of $\Sigma_{-i}^{\mathrm{U},k-1}$, which proves (A.1) for k.

Part 2: *Bidders do not bid lower than* z_U *if they play rationalizable strategies:*

$$\sigma\big([0, z_U]\big) = 0 \quad \textit{for all} \quad \sigma \in \Sigma^U.$$

Recall that

$$I_k = [(k-1)\cdot\delta, k\cdot\delta) \quad \text{for} \quad k = 1, 2, \ldots N; \quad I_0 = \emptyset,$$

and let

$$J_k = \bigcup_{l=0}^{k} I_l = [0, k\delta).$$

We will iteratively show that

$$\Sigma_i^{U,k} \subseteq \{\sigma_i \mid \sigma_i(J_k) = 0\}, \quad \text{for} \quad k = 1, 2, .., N; i = 1, 2, \tag{A.2}$$

which is sufficient to prove the Lemma. Observe that (A.2) trivially holds for $k = 0$. Assume that it holds for $k - 1 < N$ for player $-i$. We will show that

$$\Sigma_i^{U,k} \subseteq \{\sigma_i \mid \sigma_i(J_k) = 0\}. \tag{A.3}$$

Assume on the contrary

$$\exists \sigma_i \in \Sigma_i^{U,k} \quad \text{with} \quad \sigma_i(I_k) > 0. \tag{A.4}$$

We will now demonstrate that for each $\sigma_{-i} \in \operatorname{conv}\Sigma_{-i}^{U,(k-1)}$ there exists $\hat{\sigma}_i$ such that $\mathfrak{R}_i^U(\hat{\sigma}_i, \sigma_{-i}) \geq \mathfrak{R}_i^U(\sigma_i, \sigma_{-i})$. This will pose a contradiction to the above assumption (A.4), namely that σ_i is a best response to some mixed strategy from the set $\operatorname{conv}\Sigma_{-i}^{U,(k-1)}$.

Case 1: $\sigma_{-i}(J_{2k}) > 0$.

Consider the strategy $\hat{\sigma_i}$:

$$\hat{\sigma_i}(B) = \sigma_i(B \cap \complement I_k) + \sigma_i(I_k) \cdot \mathbb{1}_{k\delta \in B} \quad \text{for } B \in \mathscr{B},$$

where $\complement I_k$ is the complement set of I_k ($\complement I_k \equiv M \backslash I_k$):

$$\begin{aligned}
&\mathfrak{R}_i^U(\hat{\sigma}_i, \sigma_{-i}) - \mathfrak{R}_i^U(\sigma_i, \sigma_{-i}) \\
&\quad \geq \int \Big(\int R_i^{\mathrm{U}}(x_i, x_{-i}) \mathrm{d}\hat{\sigma}_i(x_i) - \int R_i^{\mathrm{U}}(x_i, x_{-i}) \mathrm{d}\sigma_i(x_i) \Big) \mathrm{d}\sigma_{-i}(x_{-i}) \\
&\quad = \int \Big(R_i^{\mathrm{U}}(k\delta, x_{-i}) \cdot \sigma_i(I_k) - \int_{I_k} R_i^{\mathrm{U}}(x_i, x_{-i}) \mathrm{d}\sigma_i(x_i) \Big) \mathrm{d}\sigma_{-i}(x_{-i}) \\
&\quad = \int \int_{I_k} \Big(R_i^{\mathrm{U}}(k\delta, x_{-i}) - R_i^{\mathrm{U}}(x_i, x_{-i}) \Big) \mathrm{d}\sigma_i(x_i) \mathrm{d}\sigma_{-i}(x_{-i}) && \text{(A.5)} \\
&\quad = \int_{\complement J_{(k-1)}} \int_{I_k} \Big(R_i^{\mathrm{U}}(k\delta, x_{-i}) - R_i^{\mathrm{U}}(x_i, x_{-i}) \Big) \mathrm{d}\sigma_i(x_i) \mathrm{d}\sigma_{-i}(x_{-i}) && \text{(A.6)} \\
&\quad > 0 && \text{(A.7)}
\end{aligned}$$

(A.6) follows from (A.5) because we assumed that (A.2) holds for $(k-1) < N$ for player $-i$. Further, from Theorem 2 follows that

$$\begin{aligned}
R_i^{\mathrm{U}}(x_i, x_{-i}) &< R_i^{\mathrm{U}}(k\delta, x_{-i}) \quad \text{if} \quad x_i \in I_k \quad \text{and} \quad (k-1) \cdot \delta \leq x_{-i} < 2k\delta, \\
R_i^{\mathrm{U}}(x_i, x_{-i}) &\leq R_i^{\mathrm{U}}(k\delta, x_{-i}) \quad \text{if} \quad x_i \in I_k \quad \text{and} \quad (k-1) \cdot \delta \leq x_{-i}.
\end{aligned}$$

As by assumption $\sigma_{-i}(J_{2k}) > 0$ the inequality (A.7) is also valid.

Case 2: $\sigma_{-i}(J_{2k}) = 0$. For the strategy $\hat{\sigma_i}$, where

$$\hat{\sigma_i}(B) = \sigma_i(B \cap \complement I_k) + \sigma_i(I_k) \cdot \mathbb{1}_{\frac{3v}{4} \in B} \quad \text{for } B \in \mathscr{B}$$

we observe that

$$\begin{aligned}
&\mathfrak{R}_i^U(\hat{\sigma}_i, \sigma_{-i}) - \mathfrak{R}_i^U(\sigma_i, \sigma_{-i}) \\
&\quad = \int_{\complement J_{2k}} \int_{I_k} \Big(R_i^{\mathrm{U}}(\frac{3v}{4}, x_{-i}) - R_i^{\mathrm{U}}(x_i, x_{-i}) \Big) \mathrm{d}\sigma_i(x_i) \mathrm{d}\sigma_{-i}(x_{-i}) \\
&\quad = \int_{\complement J_{2k}} \int_{I_k} \Big(R_i^{\mathrm{U}}(\frac{3v}{4}, x_{-i}) \Big) \mathrm{d}\sigma_i(x_i) \mathrm{d}\sigma_{-i}(x_{-i}) > 0. && \text{(A.8)}
\end{aligned}$$

The inequality (A.8) holds because $R_i^{\mathrm{U}}(\frac{3v}{4}, x_{-i}) > 0$ for $x_{-i} \in [0, v]$.

Proof of Lemma 1: $R_i^{\mathrm{U}}(\mathbf{x})$ *is continuous in* $\mathbf{x}$.

Let $\mathbf{x}$ be an arbitrary bid vector. We will show that for any sequence of bid vectors $\mathbf{x}^{(k)}$, $k = 1, 2, \ldots$, with $\mathbf{x}^{(k)} \to \mathbf{x}$ we have $R_i^{\mathrm{U}}(\mathbf{x}^{(k)}) \to R_i^{\mathrm{U}}(\mathbf{x})$. Using the (easy to prove) inequality

$$|a'b'c' - abc| \leq |a' - a| \cdot b'c' + a \cdot |b' - b| \cdot c' + ab \cdot |c' - c|,$$

which holds for arbitrary nonnegative reals a, b, c, a', b', c', we obtain

$$\begin{aligned}
&\left|R_i^{\mathrm{U}}(\mathbf{x}^{(k)}) - R_i^{\mathrm{U}}(\mathbf{x})\right| \\
&\leq \sum_{q=0}^{n} \left|\tau_q(\mathbf{x}^{(k)}) - \tau_q(\mathbf{x})\right| \cdot P(q; \mathbf{x}^{(k)}) \cdot \mathbb{1}_{\{\varphi_{\mathbf{x}^{(k)}}(i) \leq q\}} \\
&\qquad + \sum_{q=0}^{n} \big(v - \tau_q(\mathbf{x})\big) \cdot |P(q; \mathbf{x}^{(k)}) - P(q; \mathbf{x})| \cdot \mathbb{1}_{\{\varphi_{\mathbf{x}^{(k)}}(i) \leq q\}} \\
&\qquad + \sum_{q=0}^{n} \big(v - \tau_q(\mathbf{x})\big) \cdot P(q; \mathbf{x}) \cdot |\mathbb{1}_{\{\varphi_{\mathbf{x}^{(k)}}(i) \leq q\}} - \mathbb{1}_{\{\varphi_{\mathbf{x}}(i) \leq q\}}|.
\end{aligned}$$

This inequality can be interpreted as a decomposition of the change in expected payoff of bidder i into a *price effect*, a *quantity effect* and an *allocation effect*. As sums, differences, products, quotients, minimums and maximums of continuous functions are continuous, so are the functions $c_q^-(\cdot)$, $c_q^+(\cdot)$, $P(q; \cdot)$, and therefore

$$\left|\tau_q(\mathbf{x}^{(k)}) - \tau_q(\mathbf{x})\right| \to 0, \quad |P(q; \mathbf{x}^{(k)}) - P(q; \mathbf{x})| \to 0$$

for $k \to \infty$, which means price and quantity effect tend to 0. To complete the proof, we will now show that the allocation effect also tends to 0. This effect can be expressed as

$$\sum_{q \in L_{\mathbf{x}}} \big(v - \tau_q(\mathbf{x})\big) \cdot P(q; \mathbf{x}) \cdot |\mathbb{1}_{\{\varphi_{\mathbf{x}^{(k)}}(i) \leq q\}} - \mathbb{1}_{\{\varphi_{\mathbf{x}}(i) \leq q\}}|,$$

where

$$L_{\mathbf{x}} = \{q \mid \tau_q(\mathbf{x}) > \tau_{q+1}(\mathbf{x})\}$$

because $P(q; \mathbf{x}) = 0$ for $q \notin L_{\mathbf{x}}$.[15] In words, one needs to sum only over the positions in the announced demand curve for which an increase in quantity leads to a decrease in the stopout price. This holds true because if several bids are equal, the seller serves with a probability of one either none or all of them. Observe now that there exists k_0, such that for all $k \geq k_0$ we have:

$$x_j^{(k)} < x_i^{(k)} \text{ if } x_j < x_i \text{ and } x_j^{(k)} > x_i^{(k)} \text{ if } x_j > x_i \text{ for all } i, j \in \{1, \ldots, n\}. \tag{A.9}$$

Then the inequalities

$$\varphi_{\mathbf{x}}(i) \leq q \quad \text{and} \quad \varphi_{\mathbf{x}^{(k)}}(i) \leq q$$

[15] One observes that $c_q^- = \tau_q$ and $c_q^+ \leq \tau_q$. Hence $P(q; \mathbf{x}) = 0$.

are equivalent for $q \in L_{\mathbf{x}}$ and $k \geq k_0$, which completes the proof.

B Appendix

In this Appendix we prove Theorem 6: in every symmetric mixed strategy equilibrium of the uniform price auction buyers bid with probability one higher than z_{D} (the equilibrium bid in the discriminatory auction). First we provide some auxiliary statements in the form of several Lemmas.

Lemma 2. *Let* $\mathcal{L}(\mathbf{x}_{-i}) := \big[0;\ \min(\{x_j \mid j \neq i\} \cup \{z_{\mathrm{D}}\})\big)$.

(i) For any i and any given $\mathbf{x}_{-i}$, the partial derivative $\partial_i R_i(x_i; \mathbf{x}_{-i})$ exists in all but finitely many points $x_i \in \mathcal{L}(\mathbf{x}_{-i})$.

(ii) The partial derivative of the bidder that submitted the lowest bid is nonnegative if that bidder submitted a bid not higher than z_{D}. Formally, for any $\mathbf{x}_{-i}$

$$\partial_i R_i^{\mathrm{U}}(x_i; \mathbf{x}_{-i}) \geq 0,$$

for all $x_i \in \mathcal{L}(\mathbf{x}_{-i})$ for which $\partial_i R_i^{\mathrm{U}}$ exists.

(iii) The partial derivative of the bidder that submitted the lowest bid is uniformly bounded away from 0 if that bidder submitted a bid not higher than z_{D} and is served with positive probability. Formally, there exists $\overline{\partial} > 0$ such that for any $\mathbf{x}_{-i}$

$$\partial_i R_i^{\mathrm{U}}(x_i; \mathbf{x}_{-i}) > \overline{\partial}$$

for all $x_i \in \mathcal{L}(\mathbf{x}_{-i})$ for which $\partial_i R_i^{\mathrm{U}}$ exists and $c_i^+(x_i; \mathbf{x}_{-i}) > 0$.

Proof. **(i)** The expected payoff of bidder i is given by

$$R_i^{\mathrm{U}}(\mathbf{x}) = (v - x_i) F\big(c_n^+(\mathbf{x})\big).$$

As, by assumption, F is differentiable, we only have to show the differentiability of c_n^+. Observe that

$$c_n^+(\mathbf{x}) = \min_{0 \leq q < n} \frac{q\tau_q(\mathbf{x}) - n x_i}{q - n}$$

and define

$$\hat{q}(x_i; \mathbf{x}_{-i}) := \min\ \arg\min_{0 \leq q < n} \frac{q\tau_q(\mathbf{x}) - n x_i}{q - n},$$

then

$$c_n^+(\mathbf{x}) = \frac{\hat{q}(\mathbf{x})\tau_{\hat{q}(\mathbf{x})}(\mathbf{x}) - n x_i}{\hat{q}(\mathbf{x}) - n}.$$

We will now show that $\hat{q}(x_i; \mathbf{x}_{-i})$ is almost everywhere differentiable in x_i and as a consequence so will be $c_n^+(x_i; \mathbf{x}_{-i})$. Since $\hat{q}(x_i; \mathbf{x}_{-i})$, as a function of x_i, takes only finitely many integer values, monotonicity will be sufficient for it to be piecewise constant and therefore differentiable in all but finitely many points. So, to complete the proof, we will show that $\hat{q}(x_i; \mathbf{x}_{-i})$ is weakly decreasing in x_i. Take x_i', x_i'' with $x_i' < x_i''$, let $q' := \hat{q}(x_i'; \mathbf{x}_{-i})$ and $q'' := \hat{q}(x_i''; \mathbf{x}_{-i})$, and assume by contradiction that $q' < q''$. Observe that according to the Definition of $\hat{q}(x_i; \mathbf{x}_{-i})$ the quantity q' minimizes the quotient

$$\frac{q\tau_q(x_i', \mathbf{x}_{-i}) - nx_i'}{q - n},$$

and the quantity q'' minimizes the quotient

$$\frac{q\tau_q(x_i'', \mathbf{x}_{-i}) - nx_i''}{q - n}.$$

Considering the inequalities

$$\begin{aligned}
&\frac{q''\tau_{q''}(x_i'', \mathbf{x}_{-i}) - nx_i''}{q'' - n} - \frac{q'\tau_{q'}(x_i'', \mathbf{x}_{-i}) - nx_i''}{q' - n} \\
&= \frac{q''\tau_{q''}(x_i'', \mathbf{x}_{-i}) - nx_i''}{q'' - n} - \frac{q''\tau_{q''}(x_i'', \mathbf{x}_{-i}) - nx_i'}{q'' - n} + \frac{q''\tau_{q''}(x_i'', \mathbf{x}_{-i}) - nx_i'}{q'' - n} - \frac{q'\tau_{q'}(x_i'', \mathbf{x}_{-i}) - nx_i''}{q' - n} \\
&\geq \frac{q''\tau_{q''}(x_i'', \mathbf{x}_{-i}) - nx_i''}{q'' - n} - \frac{q''\tau_{q''}(x_i'', \mathbf{x}_{-i}) - nx_i'}{q'' - n} + \frac{q'\tau_{q'}(x_i'', \mathbf{x}_{-i}) - nx_i'}{q' - n} - \frac{q'\tau_{q'}(x_i'', \mathbf{x}_{-i}) - nx_i''}{q' - n} \\
&= \frac{n}{n - q''}(x_i'' - x_i') - \frac{n}{n - q'}(x_i'' - x_i') \\
&> 0,
\end{aligned}$$

we reach a contradiction to the statement that q'' minimizes the quotient

$$\frac{q\tau_q(x_i'', \mathbf{x}_{-i}) - nx_i''}{q - n}.$$

(ii) For all $c_i^+(x_i; \mathbf{x}_{-i}) \leq 0$, we have $R_i^{\mathrm{U}}(x_i; \mathbf{x}_{-i}) = 0$ and thus $\partial_i R_i^{\mathrm{U}}(x_i; \mathbf{x}_{-i}) = 0$. For the case $c_i^+(x_i; \mathbf{x}_{-i}) > 0$ see the next part.

(iii) By assumption there are bids strictly higher than x_i, therefore $\hat{q}(\mathbf{x}) \geq 1$ and $c_n^+(\mathbf{x}) < x_i$. Let $\underline{f} := \min_{c \in [0, \bar{c}]} f(c)$. As f is continuous and strictly positive in the interval $[0, \bar{c}]$ we have $\underline{f} > 0$. Recall also that F/f is increasing by assumption. Because $c_n^+(\mathbf{x}) < x_i < z_{\mathrm{D}}$, the following (in)equalities hold for all points in which the partial derivative exists:

$$\begin{aligned}\partial_i R_i^{\mathrm{U}}(\mathbf{x}) &= (v-x_i)\cdot f\big(c_n^+(\mathbf{x})\big)\cdot \partial_i c_n^+(\mathbf{x}) - F\big(c_n^+(\mathbf{x})\big)\\ &= (v-x_i)\cdot f\big(c_n^+(\mathbf{x})\big)\cdot \frac{n}{n-\hat{q}(\mathbf{x})} - F\big(c_n^+(\mathbf{x})\big)\\ &\geq (v-x_i)\cdot f\big(c_n^+(\mathbf{x})\big)\cdot \frac{n}{n-1} - F\big(c_n^+(\mathbf{x})\big)\\ &= \frac{n}{n-1}\cdot f\big(c_n^+(\mathbf{x})\big)\Big(v-x_i-\frac{n-1}{n}\cdot\frac{F\big(c_n^+(\mathbf{x})\big)}{f\big(c_n^+(\mathbf{x})\big)}\Big)\\ &> \frac{n}{n-1}\cdot \underline{f}\cdot\Big(v-z_{\mathrm{D}}-\frac{n-1}{n}\cdot\frac{F(z_{\mathrm{D}})}{f(z_{\mathrm{D}})}\Big) =: \overline{\partial}.\end{aligned}$$

Observe that $\overline{\partial} > 0$ because

$$v-z_{\mathrm{D}}-\frac{n-1}{n}\cdot\frac{F(z_{\mathrm{D}})}{f(z_{\mathrm{D}})} > v-z_{\mathrm{D}}-\frac{F(z_{\mathrm{D}})}{f(z_{\mathrm{D}})} = 0.$$

□

Lemma 3. *For any $\mathbf{x}\in[0,v]^n$ and any $\varepsilon>0$ for which $x_i+\varepsilon\leq v$ the following inequality holds:*

$$R_i(x_i+\varepsilon;\mathbf{x}_{-i}) - R_i(x_i;\mathbf{x}_{-i}) \geq -1\cdot\varepsilon.$$

Proof. The inequality applies because an increase in the bid of bidder i can lead to an increase in the stop-out price (with some probability), but does not lower the winning chances of that bidder. □

Lemma 4. *Let $\mathbf{x}$ be such that there exists x with $x\leq x_j<\frac{n}{n-1}\cdot x$ for all j. Then $c_n^+(\mathbf{x})>0$ (that means, the bidder with the lowest bid is served with positive probability).*

Proof. We have

$$c_n^+(\mathbf{x}) > \frac{nx-(n-1)\cdot\frac{n}{n-1}\cdot x}{n-n+1} = 0.$$

□

Lemma 5. *If all bidders except one (say, bidder i) submit a bid of $x\in[0,v)$ (that means, $x_j=x$ for $j\neq i$) then there exist $\varepsilon>0$ and $\widetilde{\partial}>0$ such that for $x_i\in[x,x+\varepsilon)$ the following inequality holds:*[16] $\partial_i R_i^{\mathrm{U}}(x_i,x,\ldots,x) > \widetilde{\partial}$.

Proof. From

$$R_i^{\mathrm{U}}(x_i,x,\ldots,x) = (v-x)\cdot F\big(\frac{nx-x_i}{n-1}\big) + (v-x_i)\cdot\Big(F(x_i)-F\big(\frac{nx-x_i}{n-1}\big)\Big)$$

[16] For $x_i=x$ we mean the derivative from above.

we obtain the partial derivative function

$$\partial_i R_i^{\mathrm{U}}(x_i, x, \dots, x) = -\frac{v-x}{n-1} \cdot f\big(\frac{nx - x_i}{n-1}\big) - \left(F(x_i) - F\big(\frac{nx - x_i}{n-1}\big)\right) + (v - x_i) \cdot \left(f(x_i) + \frac{1}{n-1} f\big(\frac{nx - x_i}{n-1}\big)\right),$$

which is continuous in x_i. As $\partial_i R_i^{\mathrm{U}}(x, x, \dots, x) = (v - x) \cdot f(x) > 0$, there exist $\varepsilon > 0$ and $\widetilde{\partial} > 0$ such that $\partial_i R_i^{\mathrm{U}}(x_i, x, \dots, x) > \widetilde{\partial}$ for $x_i \in [x, x + \varepsilon)$. □

Lemma 6. *Let* $\mathbf{x}$ *be such that there exist bidders* i, j, k *with* $x_i \geq x_k$ *and*

$$x_j > x_k + (n-1)^2 (x_i - x_k).$$

Then

$$\partial_i^+ R_i^{\mathrm{U}}(\mathbf{x}) = 0.$$

Proof. Observe that $\varphi_{\mathbf{x}}(i) \leq (n-1)$.[17] We have

$$c^+_{\varphi_{\mathbf{x}}(i)}(\mathbf{x}) \leq \frac{x_i \cdot (n-1) - x_j}{n-2}, \quad c^-_{\varphi_{\mathbf{x}}(i)}(\mathbf{x}) \geq n \cdot x_k - (n-1) \cdot x_i.$$

The identities

$$\begin{aligned} \frac{x_i \cdot (n-1) - x_j}{n-2} &< n \cdot x_k - (n-1) \cdot x_i \Leftrightarrow \\ x_i \cdot (n-1) - x_j &< n \cdot (n-2) \cdot x_k - (n-2) \cdot (n-1) \cdot x_i \Leftrightarrow \\ x_j &> (n-1)^2 \cdot x_i - n \cdot (n-2) \cdot x_k \Leftrightarrow \\ x_j &> (n-1)^2 \cdot x_i - [(n-1)^2 - 1] \cdot x_k \Leftrightarrow \\ x_j &> x_k + (n-1)^2 (x_i - x_k), \end{aligned}$$

verify that

$$c^+_{\varphi_{\mathbf{x}}(i)}(\mathbf{x}) < c^-_{\varphi_{\mathbf{x}}(i)}(\mathbf{x}) \text{ for } x_j > x_k + (n-1)^2 (x_i - x_k),$$

which completes the proof. With these preliminaries we can prove now Theorem 6. □

Proof of Theorem 6.

Let $\boldsymbol{\sigma}^*$ be a symmetric equilibrium, and let $z_* = \max\{z \mid \sigma_i^*([z, v]) = 1\}$ be the lower bound of the support of the bidders' strategies in that equilibrium. Assume by contradiction $z_* \leq z_{\mathrm{D}}$. Take an arbitrary bidder i and consider a deviation strategy σ_i^ε, which

[17] If bidder i submits also a bid of x_k, we choose $\varphi_{\mathbf{x}}$ so that bidder i obtains a number lower than bidder k.

only shifts the probability mass of the small interval $Z_*^\varepsilon = [z_*, z_* + \varepsilon)$ to the point $z_* + \varepsilon$:

$$\sigma_i^\varepsilon(B) = \sigma_i\big(B \cap \complement Z_*^\varepsilon\big) + \sigma_i\big(Z_*^\varepsilon\big) \cdot \mathbb{1}_{\{z_*+\varepsilon \in B\}} \quad \text{for } B \in \mathscr{B}.$$

We will show that, for ε small enough, this deviation strategy will be more profitable for player i, a contradiction to the equilibrium assumption. To show this, we define the intervals

$$Z := \Big[z_*, v\Big], \qquad Z_0^\varepsilon := \Big[z_* + (n-1)^2\varepsilon, \min\{v, \frac{n}{n-1} \cdot z_*\}\Big)$$

and the sets

$$\mathcal{Z} = Z^{n-1}, \quad \mathcal{Z}^\varepsilon = \bigg(\{z_*\} \cup \Big[z_* + (n-1)^2\varepsilon, v\Big]\bigg)^{n-1},$$
$$\mathcal{Z}_0^\varepsilon = (Z_0^\varepsilon)^{n-1}, \quad \mathcal{Z}_* = \{z_*\}^{n-1}.$$

Then we break down the set $\mathcal{Z}$ into the following four sets: $\mathcal{Z} \setminus \mathcal{Z}^\varepsilon$, $\mathcal{Z}^\varepsilon \setminus (\mathcal{Z}_0^\varepsilon \cup \mathcal{Z}_*)$, $\mathcal{Z}_0^\varepsilon$ and $\mathcal{Z}_*$. In the case of $n = 3$ bidders, taken from the perspective of bidder 3, all these sets are represented in Figure 7. Consider the difference

$$\begin{aligned}
&\mathfrak{R}_i^{\mathrm{U}}(\sigma_i^\varepsilon, \boldsymbol{\sigma}_{-i}^*) - \mathfrak{R}_i^{\mathrm{U}}(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*) \\
&\quad= \int_{\mathcal{Z}\setminus\mathcal{Z}^\varepsilon} \int_{Z_*^\varepsilon} \Big(R_i^{\mathrm{U}}(z_* + \varepsilon, \mathbf{x}_{-i}) - R_i^{\mathrm{U}}(x_i, \mathbf{x}_{-i})\Big) \mathrm{d}\sigma_i^*(x_i) \mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad+ \int_{\mathcal{Z}^\varepsilon\setminus(\mathcal{Z}_0^\varepsilon\cup\mathcal{Z}_*)} \int_{Z_*^\varepsilon} \Big(R_i^{\mathrm{U}}(z_* + \varepsilon, \mathbf{x}_{-i}) - R_i^{\mathrm{U}}(x_i, \mathbf{x}_{-i})\Big) \mathrm{d}\sigma_i^*(x_i) \mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad+ \int_{\mathcal{Z}_0^\varepsilon} \int_{Z_*^\varepsilon} \Big(R_i^{\mathrm{U}}(z_* + \varepsilon, \mathbf{x}_{-i}) - R_i^{\mathrm{U}}(x_i, \mathbf{x}_{-i})\Big) \mathrm{d}\sigma_i^*(x_i) \mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad+ \int_{\mathcal{Z}_*} \int_{Z_*^\varepsilon} \Big(R_i^{\mathrm{U}}(z_* + \varepsilon, \mathbf{x}_{-i}) - R_i^{\mathrm{U}}(x_i, \mathbf{x}_{-i})\Big) \mathrm{d}\sigma_i^*(x_i) \mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}).
\end{aligned}$$

For $\varepsilon > 0$ small enough, we obtain lower bounds of the four terms by using Lemma 3 for the first term, Lemmas 2 [(i)&(ii)] and Lemma 6 for the second one, Lemmas 2 [(i)&(iii)] and Lemma 4 for the third one[18] and Lemma 5 for the fourth term, which leads us to the following (in)equalities.

[18]Lemma 4 guarantees that in the considered set $c_n^+(\mathbf{x}) > 0$; lemma 2 ensures the existence of $\overline{\partial} > 0$.

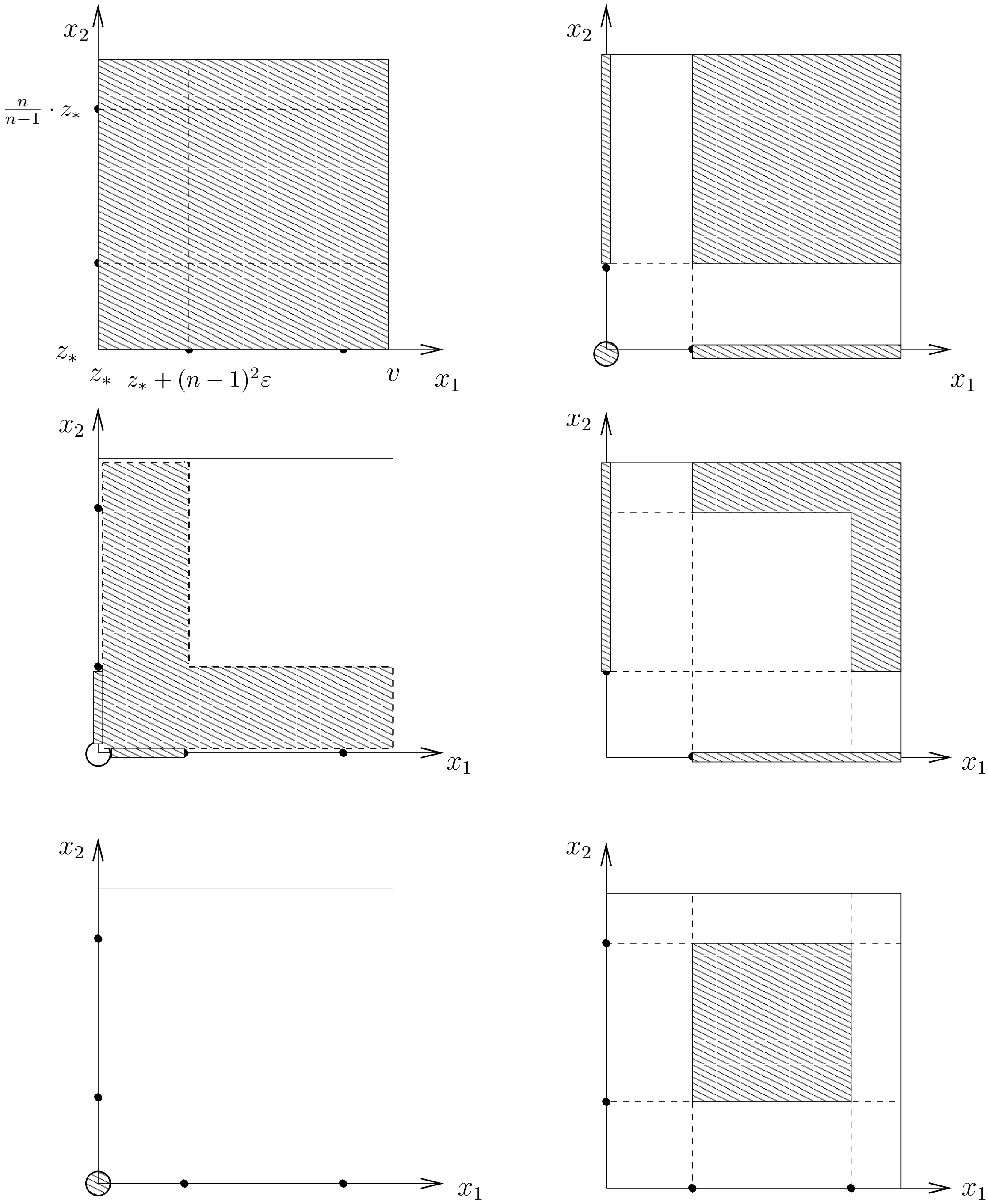

Figure 7: *The pattern ares represent the sets $\mathcal{Z}$ (upper-left), $\mathcal{Z}^\varepsilon$ (upper-right), $\mathcal{Z} \setminus \mathcal{Z}^\varepsilon$ (middle-left), $\mathcal{Z}^\varepsilon \setminus (\mathcal{Z}_0^\varepsilon \cup \mathcal{Z}_*)$ (middle-right), $\mathcal{Z}_*$ (lower-left) and Z_0^ε (lower-right) for $n = 3$ bidders.*

$$
\begin{aligned}
&\mathfrak{R}_i^{\mathrm{U}}(\sigma_i^{\varepsilon}, \boldsymbol{\sigma}_{-i}^*) - \mathfrak{R}_i^{\mathrm{U}}(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*) \\
&\qquad \geq \int_{\mathcal{Z}\setminus\mathcal{Z}^{\varepsilon}} \int_{Z_*^{\varepsilon}} (-1)\cdot(z_* + \varepsilon - x_i)\,\mathrm{d}\sigma_i^*(x_i)\mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad\qquad + \int_{\mathcal{Z}^{\varepsilon}\setminus(\mathcal{Z}_0^{\varepsilon}\cup\mathcal{Z}_*)} \int_{Z_*^{\varepsilon}} 0\cdot(z_* + \varepsilon - x_i)\,\mathrm{d}\sigma_i^*(x_i)\mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad\qquad + \int_{\mathcal{Z}_0^{\varepsilon}} \int_{Z_*^{\varepsilon}} \overline{\partial}\cdot(z_* + \varepsilon - x_i)\,\mathrm{d}\sigma_i^*(x_i)\mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad\qquad + \int_{\mathcal{Z}_*} \int_{Z_*^{\varepsilon}} \tilde{\partial}\cdot\varepsilon\,\mathrm{d}\sigma_i^*(x_i)\mathrm{d}\boldsymbol{\sigma}_{-i}^*(\mathbf{x}_{-i}) \\
&\qquad = \int_{Z_*^{\varepsilon}} (z_* + \varepsilon - x_i)\mathrm{d}\sigma_i^*(x_i)\cdot\Big((-1)\cdot\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}\setminus\mathcal{Z}^{\varepsilon}\big) + \overline{\partial}\cdot\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}_0^{\varepsilon}\big)\Big) + \tilde{\partial}\varepsilon\boldsymbol{\sigma}_{-i}^*(\mathcal{Z}_*).
\end{aligned}
$$

We will prove that for sufficiently small $\varepsilon > 0$ the expression in the last line is positive. To begin with, observe that $\lim_{\varepsilon\to 0}\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}\setminus\mathcal{Z}^{\varepsilon}\big) = 0$. So, if there exists an $\varepsilon > 0$ for which $\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}_0^{\varepsilon}\big) > 0$, then $\lim_{\varepsilon\to 0}\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}_0^{\varepsilon}\big) > 0$. Consequently, $\lim_{\varepsilon\to 0}\Big((-1)\cdot\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}\setminus\mathcal{Z}^{\varepsilon}\big) + \overline{\partial}\cdot\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}_0^{\varepsilon}\big)\Big) > 0$. If, on the other hand, $\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}_0^{\varepsilon}\big) = 0$ for all $\varepsilon > 0$, then $\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}\setminus\mathcal{Z}^{\varepsilon}\big) = 0$ and $(-1)\cdot\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}\setminus\mathcal{Z}^{\varepsilon}\big) + \overline{\partial}\cdot\boldsymbol{\sigma}_{-i}^*\big(\mathcal{Z}_0^{\varepsilon}\big) = 0$ for all $\varepsilon > 0$. In this case $\boldsymbol{\sigma}_{-i}^*(\mathcal{Z}_*) > 0$, because z_* was assumed to be the lower bound of the symmetric equilibrium mixed strategy. In either case we can state the existence of an $\varepsilon > 0$ for which the expression in the last line is positive and consequently $\mathfrak{R}_i^{\mathrm{U}}(\sigma_i^{\varepsilon}, \boldsymbol{\sigma}_{-i}^*) - \mathfrak{R}_i^{\mathrm{U}}(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*) > 0$, which completes the proof.