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RESEARCH ARTICLE

Seller competition by mechanism design

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Abstract This paper analyzes a market game in which sellers offer trading mechanisms to buyers and buyers decide which seller to go to depending on the trading mechanisms offered. In a (subgame perfect) equilibrium of this market, sellers hold auctions with an efficient reserve price but charge an entry fee. The entry fee depends on the number of buyers and sellers, the distribution of buyer valuations, and the buyer cost of entering the market. As the size of the market increases, the entry fee decreases and converges to zero in the limit. We study how the surplus of buyers and sellers depends on the number of agents on each side of the market in this decentralized trading environment.

Keywords Seller competition · Endogenous entry · Auctions

JEL Classification D44 · D82

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1 Introduction

This paper presents a model of decentralized trading in which sellers compete for a common pool of customers by offering trade mechanisms. Each seller is placed at a different location and offers one unit of a homogeneous good to buyers. Each buyer observes the available trading opportunities and decides whether to enter the market and which seller to go to. Buyer entry into the market is associated with a fixed cost, and buyers have independent private values for the good drawn from the same probability distribution. All agents are risk neutral. After going to a particular seller, each buyer learns his valuation and competes with the other buyers who visit the same seller by submitting a bid. This type of competition is akin to a variety of markets, e.g. markets for real estate, rental housing, used cars, or goods traded via the Internet. Although the model paints a highly stylized picture of trade, and abstracts from many of the institutional details of each of the aforementioned markets, it helps gain valuable insights into the nature of competition and the distribution of rents in such a decentralized trading environment.

The main purpose of this paper is to describe the trade mechanisms that sellers use in a (subgame perfect) equilibrium of this market game. We analyze equilibria in which buyers play symmetric mixed strategies by randomizing across all sellers, and equilibria in which buyers coordinate by playing pure selection strategies (i.e. choose one of the sellers or stay out of the market with a probability of one).

It is well known from the monopoly literature on auctions with endogenous entry that, when buyers make their entry decision prior to knowing their valuations, it is optimal for the seller to use a mechanism that assigns the item to the highest-valuation bidder if this valuation is higher than the seller's cost, and to keep the item otherwise (see McAfee and McMillan 1987; Engelbrecht-Wiggans 1993; Levin and Smith 1994; Chakraborty and Kosmopoulou 2001; Lu 2008). That is, the seller uses an auction with an efficient reserve price and charges an entry fee chosen so as to expropriate the entire surplus from buyers without causing them to exit the market. We identify a sufficient condition on the equilibrium behavior of buyers for which the same result applies in a setting of competing mechanism designers (Lemma 1). For this type of buyer continuation equilibria, in the first stage of the market game, sellers hold auctions with an efficient reserve price, and charge an entry fee that can vary with the number of bidders who participate in the auction (Lemma 2). There is a unique symmetric equilibrium in which sellers hold auctions with an entry fee that does not vary with the number of participants, and buyers randomize across sellers in the continuation equilibrium. We derive the entry fee in this equilibrium as a function of the number of buyers and sellers, the distribution of buyer valuations, and the buyer cost of entering the market (Proposition 1). The obtained closed form solution allows us to derive some new comparative statics results.

Perhaps most surprisingly, we find that, for certain values of the buyer cost of entry into the market, the availability of one additional seller leads to higher entry fees in equilibrium (Corollary 1). Most of the existing literature either considers the case of two sellers (see e.g. Burguet and Sakovics 1999; Moldovanu et al. 2008) or posits large markets (see e.g.McAfee 1993; Peters 1997; Peters and Severinov 1997; Satterthwaite and Shneyerov 2007; Shneyerov and Wong 2010; Eeckhout and Kircher 2010), and, thus, does not focus on comparative statics issues regarding the number of participants. A notable exception is Virág (2010) who analyzes the competition among auctioneers in reserve prices (instead of entry fees) and finds, in accord with standard intuition, that the reserve price in a symmetric equilibrium increases when the ratio of buyers to sellers increases. In contrast to Virág's model, we assume that buyers incur a fixed cost to enter the market and learn their valuation after visiting one of the sellers. Further, in the present setting, sellers are not bound to using second-price auctions with a reserve price only, but can select from a larger class of trade mechanisms.

The equilibrium entry fees do not change in the expected direction for two reasons. First, when sellers compete by setting entry fees, they earn revenue from two sources: the entry fees and the proceeds from selling the item in the auction. When one additional seller offers an item in the market, the competition among buyers at each of the sellers' auctions decreases, and the entry fees become a more important source of revenue. To account for the lower revenues in their auctions, sellers find it profitable to collectively raise their entry fees in equilibrium. Second, buyers' expenditures come also from two sources: the cost of entering the market, and the amount paid to the seller. When one additional seller is available in the market, the total surplus that a buyer can realize from trade increases and buyers find it more worthwhile to enter the market. Thus, sellers can raise the entry fees in equilibrium without causing buyers to exit the market.

The result that sellers do not set reserve prices (but charge positive entry fees instead) in their equilibrium auctions is tied to the assumption that buyers learn their valuations after visiting a particular seller. In a monopoly model, Lu (2009) analyzes the alternative scenario in which buyers know their valuation prior to deciding whether to enter the auction. A revenue-maximizing mechanism in this monopoly setting can be a second-price auction with a property set (positive) reserve price and an entry subsidy. Peck (1996) presents a model of competing firms in which a set of identical consumers arrive at the market randomly and shows that the posted price is the only possible transaction mechanism used by sellers in equilibrium.

The closed form solution we obtain for the equilibrium entry fee allows us to examine the equilibrium trade mechanisms in large markets. When the number of buyers and sellers increases, but their ratio remains the same (i.e. the thickness of the market increases), the entry fee decreases, and in the limit converges to zero (Proposition 2). The expected surplus of a buyer is affected less by the deviation of a single seller, the more agents there are in the market. Thus, the higher the number of sellers, the higher will be the outflow of buyers (in probabilistic terms) that a single seller will see if this seller raises his entry fee. Because of this effect, the equilibrium entry fee converges to zero when the number of agents goes to infinity.

The explicit solution for the equilibrium entry fee also allows us to further analyze the division of the trade surplus between buyers and sellers in some special cases. In particular, we analyze the cases in which (a) buyer values are deterministic and (b) buyer values are uniformly distributed, assuming that the buyer cost of entering the market is zero. In both cases, the buyer share of the surplus grows with the size of the market, and when the number of agents converges to infinity, buyers earn more than sellers. Buyers earn a larger share of the surplus when their values are uniformly distributed, compared to the case of deterministic valuations, which suggests that buyers are able to extract an informational rent when their valuations are private. We also analyze equilibria in which sellers use entry fees that can vary with the number of buyers. In a market with two buyers and two sellers only, we show that multiple equilibria exist, and any division of the surplus between buyers and sellers can arise as an equilibrium outcome (Proposition 2).

Further, we focus on equilibria in which buyers coordinate among the sellers by playing pure selection strategies (i.e. they select which seller to go to in a deterministic way). In this case, equilibria exist only if the buyers' cost of entering the market is sufficiently large, and in all equilibria, sellers expropriate the entire surplus from buyers (Proposition 3). Moldovanu et al. (2008) consider a related framework in which buyers learn their valuations after choosing a seller and play pure selection strategies. In contrast to the present setting, they consider only two sellers, who simultaneously choose how many units to produce and sell in uniform price auctions (instead of choosing trading mechanisms). In line with the result presented here, they find that, when the marginal production cost of sellers is small, the market game has no non-trivial equilibria (i.e. equilibria in which sellers choose positive quantities and make positive profits).

2 The model

There is a finite number of sellers indexed by $j \in J$ and a finite number of buyers indexed by $i \in I$. We will use the notation J (I) to designate both the set and number of sellers (buyers). Each seller has one unit to sell, and each buyer seeks to buy one unit of a homogeneous good. All agents are risk neutral. The use value of the good is the same for each seller, and without loss of generality is normalized to zero. The trading process is described by the following two-stage game. First, sellers simultaneously offer transaction mechanisms to buyers, and then, upon observing the available trade mechanisms, buyers decide whether to enter the market, and which seller to go to. Entry into the market is associated with a fixed cost of $c \geq 0$ for each buyer. When buyers decide which seller to visit, they know the transaction mechanism advertised by each seller, and the distribution function F of their private valuation for the goods. Upon visiting a particular seller, buyers privately observe their valuations and the number of other buyers who go to that seller. Then, buyers submit bids to the seller, the mechanisms are operated, and the transactions take place. Thus, the model we consider here is similar to the monopoly model by Levin and Smith (1994) and differs from the model developed by Samuelson (1985) who assumes that buyers know their valuations before entering the market. For empirical tests of alternative models of entry, see Marmer et al. (2007). Our discussion focuses on the symmetric independent private value setting: valuations are symmetrically and independently distributed according to a continuously differentiable distribution function F with support normalized to [0, 1]. A mechanism prescribes an allocation and a payment for any number (and identity) of bidders and any realization of their valuations. Let us denote the set of the subsets (the power set) of all bidders by $\mathcal{I}$ and the power set of all rivals of bidder i by $\mathcal{I}^{-i}$. Let $s \in \mathcal{I}$ denote a group of bidders and let x^s be the ordered vector[1]

[1] By ordered vector x^s we refer to the vector of valuations of the bidders from a subset s, in which the components are ordered in an ascending order according to the bidder's number.

of their valuations. Further, let X^s denote the set of all possible ordered vectors of valuations of the bidders from group s. We denote by $X \equiv \bigcup_{s \in \mathcal{I}} X^s$ the set of all ordered vectors of the valuations of all subsets of bidders and by x an element of this set.[2] Similarly, X_{-i} denotes the set of ordered vectors of the valuations of all subsets of bidders that do not contain bidder i.

2.1 Seller strategy space (trade mechanisms)

A direct mechanism consists of an *allocation rule*, $p_i : X \to [0, 1]$, and a *payment rule*, $z_i : X \to \mathbb{R}$. The allocation rule maps the bidders' reports of their valuations (bids) into a probability of winning for each bidder $i \in I$, and the payment rule maps the bidders' messages into an expected payment of each bidder i to the seller. We denote the strategy set of sellers by

$$\mathfrak{A} \equiv \{\{(p_i, z_i)\}_{i \in I} \mid (p_i, z_i) : X \to [0, 1] \times \mathbb{R}\}$$

and require the allocation and payment rules to satisfy conditions (P), (F), (AN), and (IC) given below. The mechanism of seller j is denoted by A_j, and the vector of the mechanisms of all sellers by $A = (A_1, A_2, \ldots, A_J)$.

- (P) *Participation.* $p_i(x^s) = 0,\ z_i(x^s) = 0,\ \forall i \notin s$. Only buyers who participate in a certain mechanism can win the object and be required to pay.
- (F) *Feasibility.* $\sum_{i=1}^{I} p_i(x) \leq 1, \forall x \in X$. The allocation rule does not allow more units to be sold than are physically available for any realization of buyer valuations.[3]
- *(AN) Anonymity*. Sellers do not discriminate among buyers on characteristics different than their bids. In other words, the chances of winning and the payment cannot depend on the buyers' identities but solely on their bids. Formally, the functions p and z are required to be *permutation invariant*. This means permuting the valuations of any ordered vector $x \in X$ permutes the vectors $p(x)$ and $z(x)$ in the same fashion.
- *(IC) Incentive compatibility:*
  Assume bidder i participates in mechanism A_j. At the time of bidding, he knows his valuation x_i and the set of bidders who participate in the same mechanism. Because we require mechanisms to be anonymous, the probability of winning the item and the payment of bidder i depend only on the number of his rivals, and not on their identity. Let the number of all bidders visiting seller j be n. If bidder i reports the valuation $\tilde{x}_i$, and all other bidders report truthfully, the expected

[2] Note that the valuation of each bidder i, x_i, might or might not appear in the vector x depending on whether this bidder participates in the mechanism or not.

[3] For some realizations of x, the strict inequality may hold. The strict inequality will hold, for instance, when the mechanism is a second-price auction with a positive reservation price. In this case, if the valuation of the participating bidders is below the seller's reserve price, the seller will not sell the item.

probability of winning the item and the expected payment will respectively be

$$P_{A_j}^{(n)}(\tilde{x}_i) := \int p_i(\tilde{x}_i, x^{(n-1)}) \mathrm{d}F(x^{(n-1)}),$$
$$Z_{A_j}^{(n)}(\tilde{x}_i) := \int z_i(\tilde{x}_i, x^{(n-1)}) \mathrm{d}F(x^{(n-1)}),$$

where $x^{(n-1)}$ is the vector of other bidders' bids, $p_i(\tilde{x}_i, x^{(n-1)})$ is the allocation rule, and $z_i(\tilde{x}_i, x^{(n-1)})$ is the payment rule specified in mechanism A_j. The incentive compatibility condition requires that bidder i finds it profitable to report truthfully if all other bidders do so. Hence, for every n and every $\tilde{x}_i \in [0, 1]$, the inequality

$$\begin{aligned} E_{A_j}^{(n)}(\tilde{x}_i \mid x_i) &=: x_i \cdot P_{A_j}^{(n)}(\tilde{x}_i) - Z_{A_j}^{(n)}(\tilde{x}_i) \\ &\leq x_i \cdot P_{A_j}^{(n)}(x_i) - Z_{A_j}^{(n)}(x_i) \equiv E_{A_j}^{(n)}(x_i \mid x_i) =: E_{A_j}^{(n)}(x_i) \end{aligned}$$

holds. Hereby, $E_{A_j}^{(n)}(\tilde{x}_i \mid x_i)$ denotes the expected payoff of a bidder, who has a valuation of x_i, reports the valuation $\tilde{x}_i$ to seller j, and faces $(n-1)$ competing bidders.

The restriction that sellers use direct mechanisms is not without loss of generality, since additional equilibria may arise. As buyers observe the mechanisms offered by all sellers, each seller can potentially ask bidders to report the mechanisms offered by the other sellers and condition the allocation of the item on these reports. Here, we ruled out this possibility as has been done in most of the literature on competing mechanisms (see e.g. McAfee 1993; Peters 1997). We note that the equilibrium allocation derived here will still be an equilibrium outcome even if sellers can make the allocation dependent on the mechanisms used by other sellers. In equilibrium, sellers know the mechanisms used by their competitors and there is no benefit to a seller from making the allocation dependent on the mechanisms of other sellers if no other seller does so. Allowing for a larger strategy space for sellers, however, may lead to additional equilibria (see Epstein and Peters 1999; Peters 2001).

2.2 Buyer strategy space

Conditional on observing the mechanisms on offer, each buyer chooses which seller to go to. Buyers randomize across the set of sellers and the option to stay out of the market. A buyer's strategy, $m^i : \mathfrak{A}^J \to \Delta(J \cup 0)$ is a mapping from the set of seller trade mechanisms into the set of probability distributions over the sellers, J, and the option to staying out of the market denoted by 0.

2.3 Payoffs

When bidder i visits seller j with a probability of one, and all other bidders visit this seller with a probability of m, the expected payoff of bidder i is given by

$$R_j^i(A_j; m) = \sum_{n=1}^{I} \Pr[n-1; m] \cdot \int_0^1 E_{A_j}^{(n)}(x_i)\mathrm{d}F(x_i) - c,$$

where

$$\Pr[n-1; m] = \sum_{n=1}^{I} \binom{I-1}{n-1} m^{n-1}(1-m)^{I-n}$$

is the binomial probability with which bidder i faces $(n-1)$ rivals. If all buyers go to seller j with a probability of m, the expected payoff of seller j is

$$\Pi_j(A_j; m) = \sum_{n=1}^{I} \Pr[n; m] \cdot \int_0^1 Z_j^{(n)}(x_i)\mathrm{d}F(x_i),$$

where

$$\Pr[n; m] = \binom{I}{n} m^n (1-m)^{I-n}$$

is the probability that exactly n bidders go to seller j. We will denote the strategy of bidder i by

$$m^i(\cdot) = \left(m_0^i(\cdot), m_1^i(\cdot), m_2^i(\cdot) \ldots, m_J^i(\cdot)\right),$$

where $m_j^i(\cdot)$ is the probability that buyer i goes to seller j and $m_0^i(\cdot)$ is the probability with which buyer i does not enter the market.

In this paper, we will focus on symmetric equilibria. Following the approach by Peters and Severinov (1997) and Burguet and Sakovics (1999), we will search for (subgame perfect) equilibria in which both buyers and sellers behave symmetrically. Symmetry on the part of sellers means that sellers use the *same transaction mechanism*. Symmetry on the part of buyers means that buyers use the *same selection rule*. That is, for each set of transaction mechanisms on offer, all buyers randomize across sellers in the same fashion. Thus, in each subgame defined by sellers' transaction mechanisms, we are interested in the (mixed strategy) Nash equilibria in which all buyers choose the same probability for visiting a certain seller. Given that all other buyers use the same mixed strategy, we will look for the best response of an individual buyer. Our focus on symmetric behavior allows us to economize somewhat on notation. For the equilibrium analysis, we need only the payoff of an individual buyer depending on his own strategy, and the strategy used by the other buyers which is the same for all other buyers. Let all other bidders play the (same) mixed strategy given by

$$m^{-i}(\cdot) = \left(m_0^{-i}(\cdot), m_1^{-i}(\cdot), m_2^{-i}(\cdot) \ldots, m_J^{-i}(\cdot)\right).^4$$

If sellers play the strategy profile A, the expected payoff of bidder i is given by[5]

$$ER_i\left(A; m^i(A), m^{-i}(A)\right) = \sum_{j=1}^{J} m_j^i(A) \cdot \left(R_j^i(A_j; m_j^{-i}(A)) - c\right).$$

2.4 Equilibrium

Next, we provide a formal definition of a symmetric subgame perfect equilibrium of this decentralized trading game.

Definition 1 The seller strategy profile A^* and the bidder strategy profile $m^*(\cdot)$ constitute a (symmetric subgame perfect) equilibrium, if they satisfy the following conditions.

Buyers play a symmetric mixed strategy Nash equilibrium in every subgame A.

$$ER_i\left(A; m^{*i}(A), m^{*-i}(A)\right) \geq ER_i\left(A; m^i(A), m^{*-i}(A)\right), \quad \forall A \in \mathfrak{A}^J, \quad \forall i, \forall m^i \in [0, 1], \tag{BN}$$

$$m^{*i}(A) = m^{*-i}(A), \quad \forall A \in \mathfrak{A}^J, \quad \forall i. \tag{BS}$$

Sellers play a symmetric Nash equilibrium in the first stage of the game.

$$\Pi_j\left(A_j^*, A_{-j}^*; m^*(\cdot)\right) \geq \Pi_j\left(A_j, A_{-j}^*; m^*(\cdot)\right), \quad \forall A_j \in \mathfrak{A}. \tag{SN}$$

$$A_1^* = A_2^* = \cdots = A_J^*. \tag{SS}$$

Analyzing symmetric continuation equilibria for buyers is appealing for several reasons. First, such behavior appears quite intuitive because buyers are assumed to be ex ante identical and anonymous, and there is no device in this game that allows them

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[4] Here, $m_0^i(\cdot)$ is the probability with which bidder i does not enter the market, and $m_0^{-i}(\cdot)$ is the probability with which another bidder does not enter the market. $m_j^i(\cdot)$ and $m_j^{-i}(\cdot)$ are the probabilities, with which bidder i and another bidder, respectively, go to seller $j = 1, 2, \ldots, J$.

[5] The expected payoff of each bidder can be defined in a similar way when other bidders play asymmetric (mixed) strategies. In this case, the expected payoff $R_j^i(A_j; \cdot)$ will depend on the probabilities with which each rival of bidder i visits seller j. To conserve space, we will not introduce these payoffs here as they are not needed for the analysis of symmetric equilibria.

to coordinate. Second, the number of bidders in each auction is stochastic, and this is a phenomenon frequently observed in practice. Finally, the probability of entry changes continuously with the mechanisms' variables, e.g. entry fees or reserve prices, and, as we will see, this property ensures the existence of an equilibrium. We will characterize this type of equilibrium as a function of the characteristics of market participants. An undesirable aspect of this type of buyer behavior is the inefficiency resulting from the lack of coordination among buyers. Because buyers distribute stochastically, the units offered by sellers are not always allocated to the buyers with the highest valuations, and some items might even remain unsold because each sellers receives no buyers with positive probability. A continuation equilibrium that does not suffer from this inefficiency problem is an equilibrium in which buyers can coordinate and sort themselves among the sellers in a way that each seller receives about the same number of buyers. While this type of buyer behavior also does not ensure that the highest value bidders always get the items available, it at least guarantees that no items remain unsold. In Sect. 7, we will analyze equilibria in which buyers coordinate.

3 Analysis

In this section, we first describe the types of mechanisms that can emerge as equilibria of this decentralized market game. First, we show that, for a set of buyer continuation equilibria, the only trade mechanisms that can form an equilibrium are auctions with no reserve price but with an entry fee that might depend on the number of participating bidders (Lemmas 1 and 2). Second, we show that if we find an equilibrium in the game in which sellers hold auctions with an entry fee that is independent of the number of bidders, then this equilibrium strategy profile is also an equilibrium in the game in which sellers are allowed to vary the entry fee depending on the number of bidders attending their auction (Lemma 3). Finally, we characterize in detail an equilibrium of this market game in which sellers hold auctions with an entry fee, and we provide a closed form solution for the equilibrium entry fee (Proposition 1). For the analysis presented in Lemma 1 through Lemma 3, we will restrict attention to buyer continuation equilibria that satisfy the following property:

For all mechanisms $\tilde{A}_j$ and A_j for which

$$R_j\left(\tilde{A}_j; m_j^*(A_j, A_{-j})\right) = R_j\left(A_j; m_j^*(A_j, A_{-j})\right)$$

buyers employ the same (equilibrium) randomization strategy, i.e.

$$m_j^*(\tilde{A}_j, A_{-j}) = m_j^*(A_j, A_{-j}), \forall \tilde{A}_j, A_j, \quad \text{and} \quad A_{-j} \in \mathfrak{A}; \quad \forall j \in J. \qquad \text{(BC)}$$

Property (BC) implies that buyers do not change their probability of visiting seller j unless seller j deviates to a mechanism that provides them with a higher or a lower expected surplus. If seller j plays a deviation that offers bidders the same expected surplus, then the condition (BC) requires that the probability with which buyers visit seller j remains unchanged.[6] Note that this property rules out some continuation equilibria for buyers but does not impose non-equilibrium behavior of buyers. Our first statement narrows down the set of mechanisms that can constitute an equilibrium.

Lemma 1 *If buyers play a continuation equilibrium that satisfies condition (BC), only mechanisms that assign the item to the highest-valuation bidder can form an equilibrium in the first stage of the game.*

Proof Take an equilibrium strategy profile (A_j, A_{-j}), and let buyers visit seller j with a probability of m (i.e. $m_j^*(A_j, A_{-j}) = m$) and earn an expected payoff of $R_j\left(A_j; m\right) = R$ in the continuation equilibrium. Condition (BC) says that for all mechanisms $\tilde{A}_j$ such that $R_j\left(\tilde{A}_j; m\right) = R$, buyers visit seller j with a probability of m. Because A_j is the equilibrium strategy of seller j, among the mechanisms that give each bidder an expected payoff of R (given the equilibrium probability m with which each bidder visits seller j), the mechanism A_j maximizes the expected payoff of seller j given by

$$\Pi_j\left(A_j; m\right) = \sum_{n=1}^{I} \Pr[n; m] \cdot n \cdot \int_0^1 Z_{A_j}^{(n)}(x_i) \mathrm{d}F(x_i).$$

Since each bidder earns an expected payoff of R, the sum of the expected payoffs of all bidders is $I \cdot m \cdot R$ (each bidder goes to seller j with a probability of m and earns an expected payoff of R, and there are I buyers in total). Formally,

$$I \cdot m \cdot R = \sum_{n=1}^{I} \Pr[n; m] \cdot n \cdot \int_0^1 \left(x_i \cdot P_{A_j}^{(n)}(x_i) - Z_{A_j}^{(n)}(x_i)\right) \mathrm{d}F(x_i).$$

Thus, seller j chooses A_j to maximize his expected payoff

$$\sum_{n=1}^{I} \Pr[n; m] \cdot n \cdot \int_0^1 \left(x_i \cdot P_{A_j}^{(n)}(x_i)\right) \mathrm{d}F(x_i) - I \cdot m \cdot R$$

[6] In a setting in which sellers are constrained to using only auctions (with no entry fees), this condition is automatically satisfied because the expected payoff of each bidder $R_j(A_j; m)$ is monotonically decreasing in the probability m with which other bidders go to seller j. For the general mechanisms considered here, however, $R_j(A_j; m)$ may not be monotonic in m. This implies that multiple mixed strategy continuation equilibria for buyers may exist. Hence, when seller j deviates to another mechanism, buyers could possibly shift to another continuation equilibrium and change their probability of visiting seller j even if the deviation of seller j is revenue equivalent to them. The purpose of property (BC) is to rule out this type of (inconsistent) buyer behavior.

subject to the constraint $R_j\left(A_j; m\right) = R$. This expression reaches a maximum when

$$\int_0^1 \left(x_i \cdot P_{A_j}^{(n)}(x_i)\right) \mathrm{d}F(x_i)$$

is maximized for every $n = 1, 2, \ldots, I$. Because $x_{-i} \in [0, 1]^{n-1}$, we have

$$P_{A_j}^{(n)}(x_i) = \int_{[0,1]^{n-1}} p_i(x_i, x_{-i})\mathrm{d}F(x_{-i})$$

and since we consider only anonymous mechanisms (see condition *(AN)*), we obtain

$$\int_0^1 \left(x_i \cdot P_{A_j}^{(n)}(x_i)\right) \mathrm{d}F(x_i) = \int_{[0,1]^n} \left(\sum_{i=1}^{I} x_i \cdot p_i(x)\right) \mathrm{d}F(x).$$

This expression reaches a maximum when, for every participating bidder i, probability $p_i(x)$ is chosen so that $p_i(x) = \begin{cases} 1 & \text{if } x_i \text{ is the highest valuation,} \\ 0 & \text{otherwise.} \end{cases}$ □

Lemma 2 *If buyers play a continuation equilibrium that satisfies condition (BC), all equilibrium mechanisms (if an equilibrium exists) are expected payoff equivalent to a second-price auction with a zero reserve price and an entrance fee which might depend on the number of participating bidders.*

Proof Using condition (IC) and the Envelope Theorem, we obtain $\frac{\mathrm{d}}{\mathrm{d}\tilde{x}_i}(E_{A_j}^{(n)}(\tilde{x}_i \mid x_i) = \frac{\partial}{\partial \tilde{x}_i}(E_{A_j}^{(n)}(\tilde{x}_i \mid x_i))\big|_{\tilde{x}_i = x_i} = [F(x_i)]^{n-1}$. The expected payoff of a bidder with a valuation of x_i who faces $(n-1)$ rivals is thus

$$E_{A_j}^{(n)}(x_i \mid x_i) = (-C^n) + \int_0^{x_i} [F(z)]^{n-1}\mathrm{d}z,$$

where $(-C^n)$ is the expected payoff of the bidder with the lowest valuation ($x_i = 0$). The amount C^n is the entrance fee that each bidder has to pay when participating with $(n-1)$ other bidders. □

Thus, an equilibrium mechanism can be described by the participation fees $C^1, C^2, \ldots, C^I$ that the seller specifies for each number of bidders. So, from now on, we will say that sellers use auctions if they use a mechanism expected revenue equivalent to a second-price auction with entry fees.

Lemma 3 *Assume that buyers play a continuation equilibrium that satisfies condition (BC). Consider a strategy profile in which each seller uses an auction with an entry* *fee C that does not vary with the number of buyers (let us call it a non-variable entry fee). If this strategy profile is an equilibrium of the game in which sellers can use only non-variable entry fees (let us call it game NV), then it is also an equilibrium of the game in which sellers can use entry fees $C^1, C^2, \ldots, C^I$ that can vary with the number of participating bidders (let us call it game V).*

Proof Assume by contradiction that the strategy profile in which all sellers choose a non-variable entry fee C forms an equilibrium of the game NV, but is not an equilibrium of the game V. Then, there exist entry fees $\tilde{C}^1, \tilde{C}^2, \ldots, \tilde{C}^I$ for one of the sellers (let that be seller j), which lead to a higher expected payoff for this seller:

$$\Pi_j \left((\tilde{C}^1, \tilde{C}^2, \ldots, \tilde{C}^I); \tilde{m} \right) > \Pi_j \left((C, C, \ldots, C); m \right),$$

where $\tilde{m}$ is the equilibrium probability with which all bidders go to seller j when this seller plays $\tilde{C}^1, \tilde{C}^2, \ldots, \tilde{C}^I$, and m is the equilibrium probability when seller j uses the non-variable entry fee C. The expected payoff of each bidder i is

$$R_j^i \left(\tilde{A}_j; \tilde{m} \right) = \sum_{n=1}^{I} \Pr[n-1; \tilde{m}] \cdot B_n - \sum_{n=1}^{I} \Pr[n-1; \tilde{m}] \cdot \tilde{C}^n,$$

and of seller j is

$$\Pi_j \left(\tilde{A}_j; \tilde{m} \right) = \sum_{n=1}^{I} \Pr[n; \tilde{m}] \cdot S_n + \sum_{n=1}^{I} \Pr[n; \tilde{m}] \cdot n \cdot \tilde{C}^n.$$

Let us now construct a deviation with a non-variable entry fee

$$\tilde{C} = \sum_{n=1}^{I} \Pr[n-1; \tilde{m}] \cdot \tilde{C}^n.$$

That is, $\tilde{C}$ is chosen so that buyers pay on average the same amount in entry fees as in the mechanism $(\tilde{C}^1, \tilde{C}^2, \ldots, \tilde{C}^I)$. Property (BC) guarantees that buyers will continue to visit seller j with a probability of $\tilde{m}$ because their expected payoff has not changed. We obtain

$$\begin{aligned} \Pi_j \left((\tilde{C}, \tilde{C}, \ldots, \tilde{C}); \tilde{m} \right) &= \sum_{n=1}^{I} \Pr[n; \tilde{m}] \cdot S_n + \sum_{n=1}^{I} \Pr[n; \tilde{m}] \cdot n \cdot \tilde{C} \\ &= \sum_{n=1}^{I} \Pr[n; \tilde{m}] \cdot S_n + I \cdot \tilde{m} \cdot \tilde{C}. \end{aligned}$$

Note that the expected fees that seller j obtains with the mechanisms $(\tilde{C}^1, \tilde{C}^2, \ldots, \tilde{C}^I)$ and $(\tilde{C}, \tilde{C}, \ldots, \tilde{C})$ are equal because each bidder pays the same expected fee. It follows that

$$\Pi_j\left((\tilde{C},\tilde{C},\ldots,\tilde{C});\tilde{m}\right)=\Pi_j\left((\tilde{C}^1,\tilde{C}^2,\ldots,\tilde{C}^I);\tilde{m}\right)>\Pi_j\left((C,C,\ldots,C);m\right),$$

a contradiction. □

We note that, in addition to the equilibrium in which sellers use non-variable entry fees, other equilibria equilibrium entry fees, other equilibria exist in which sellers use different entry fees depending on the number of bidders. We will characterize these equilibria for a market with 2 buyers and 2 sellers in Sect. 6. In the next statement, we describe the equilibrium of the game NV in which sellers hold auctions with a non-variable entry fee.

Let

$$R[m]:=\sum_{n=1}^{I}\Pr[n-1;m]\cdot B_n$$

denote the payoff of a bidder who participates in an auction with no entry fee or reserve price, provided that all other bidders participate in the same auction with a probability of m. We define now

$$E:=\left[m\cdot\frac{\mathrm{d}}{\mathrm{d}m}R\left[\frac{1-m}{J-1}\right]\right]\Bigg|_{m=1/J}.$$

We will demonstrate in the next proposition that, when the buyer cost of entering the market is small, in equilibrium all sellers charge a positive entry fee equal to E, buyers visit each seller with a probability of $1/J$ and realize a trade surplus in excess of their entry cost.

Proposition 1 *There is an equilibrium in which sellers hold auctions with a non-variable entry fee. The entry fee is uniquely determined by the number of buyers and sellers, the distribution of buyer valuations, and the buyer cost of entering the market and is given as follows.*

Case 1. $c<R[1/J]-E$. *Sellers use an entry fee equal to* E. *All buyers enter the market with a probability of one and receive an expected payoff higher than their cost of entry.*

Case 2. $R[1/J]-E\le c<R[1/J]$. *All buyers enter the market with a probability of one and just cover their cost of entry. All sellers use an entry fee of* $R[1/J]-c>0$. *As the cost of entering the market increases, all sellers lower their entry fees so as to allow buyers to enter the market with a probability of one.*

Case 3. $c\ge R[1/J]$. *Buyers randomize between entering and not entering the market. The entry fee in the equilibrium auctions is zero. In particular, if* $c\ge R[0]$, *buyers stay out of the market with a probability of one.*

The solid line in Fig. 1 illustrates how the entry fee depends on the buyer cost of entering the market for a given distribution of buyers' valuations. A detailed proof of

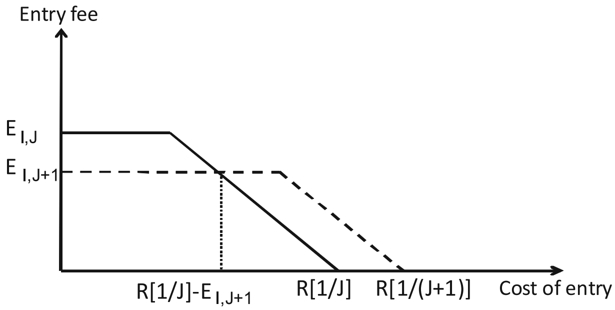

Fig. 1 Equilibrium entry fee in a market with J sellers (*firm line*) and in a market with $J+1$ sellers (*dash line*)

the proposition can be found in the Appendix. Here, we will focus on the most interesting case (Case 1) for which we will sketch the main arguments and explain how the sellers' problem relates to the literature on common agency and menu auctions. Let us denote the expected joint surplus of one seller and n bidders who participate in an auction with no entry fee and no reserve price by

$$JS_n := S_n + n \cdot B_n.$$

If all buyers visit a given seller j with a probability of m, the joint surplus of all buyers and this seller can be defined as

$$JS(m) := \sum_{n=1}^{I} \Pr[n; m] \cdot JS_n.$$

Assume now that all other sellers charge E. In equilibrium in which bidders randomize, they earn the same expected surplus with each seller. Hence, seller j leaves each buyer an expected payoff equal to the amount that buyers earn with the other sellers, which is $R[\frac{1-m}{J-1}] - E$. Thus, seller j's payoff is

$$\Pi_j(m, E) = JS(m) - I \cdot m \cdot \left(R\left[\frac{1-m}{J-1}\right] - E \right).$$

The problem of this seller is choosing the entry probability m in such a way that his profit is maximized. This problem is equivalent to the resource allocation and menu auction problem considered in Bernheim and Whinston (1986) in which bidders report for each m their cost of participating in the mechanism of auctioneer j (for each m the truthful report is $R\left[\frac{1-m}{J-1}\right] - E$), and the auctioneer seeks to implement an action $m \in [0, 1]$ that maximizes his payoff. Observe that in this setting the profit of seller j equals the total surplus generated from trade net of the buyer opportunity cost of visiting one of the other sellers. Hence, the seller's problem is one of implementing an efficient allocation. Williams (1999) shows that any efficient and incentive compatible mechanism is payoff equivalent to a Groves mechanism, so the problem of seller j can be viewed as a Groves implementation problem. Using some properties of the independent private value model, we show that $JS'(m) = I \cdot R[m]$ (Lemma 5 in the Appendix) and obtain

$$\frac{\mathrm{d}\Pi_j(m, E)}{\mathrm{d}m} = I \cdot \left[R[m] - \left(R\left[\frac{1-m}{J-1}\right] - E \right) - m \cdot \frac{\mathrm{d}}{\mathrm{d}m} R\left[\frac{1-m}{J-1}\right] \right].$$

In equilibrium $m = 1/J$, so $R[m] = R\left[\frac{1-m}{J-1}\right]$ and the first derivative simplifies to

$$\frac{\mathrm{d}\Pi_j(m, E)}{\mathrm{d}m} = I \cdot \left[E - m \cdot \frac{\mathrm{d}}{\mathrm{d}m} R\left[\frac{1-m}{J-1}\right] \right].$$

Thus, ignoring the entry fee, seller j's marginal profit from increasing the frequency m with which buyers visit him is simply -1 multiplied by the average number of buyers, $I \cdot m$, multiplied by the marginal change in the expected payoff that a buyer earns from the other sellers. In equilibrium, E offsets this change in expected marginal profit.

Proposition 1 is useful because it allow us to analyze how the competition among sellers affects the equilibrium entry fee. As the next result states, for some values of the buyer cost of entry, the equilibrium entry fee increases when one more seller joins the market.

Corollary 1 *When $R[1/J] - E_{I,(J+1)} < c < R[1/(J+1)]$, the equilibrium entry fee in a market with $J+1$ sellers and I buyers is higher compared to the entry fee in a market with J sellers and I buyers.*

The corollary follows directly from Fig. 1, which illustrates the entry fees in a market with J and $J+1$ sellers, holding the number of buyers and the distribution of their valuations fixed. The function $R[m]$ is strictly decreasing in m (see Lemma 4 in the Appendix), so $R[1/J] < R[1/(J+1)]$ and the position of the functions describing the entry fees in the two markets is as represented in Fig. 1.

4 Large markets

Following the approach by McAfee (1993) and Peters and Severinov (1997), we hold the ratio of buyers to sellers constant, $k = I/J$ and consider a sequence of markets for which $J = 1, 2, 3, \ldots$. We focus on the interesting case in which the buyer cost of entry is so low that in equilibrium no buyers stay out of the market.[7]

[7] The alternative case is trivial because, as has been shown, the equilibrium entry fee is zero.

4.1 Entry fees

Corollary 2 *As the number of buyers and sellers converges to infinity, but their ratio remains constant, i.e.* $I = k \cdot J$*, the entry fee converges to zero:*

$$\lim_{J\to\infty} E_{J,k\cdot J} = 0.$$

A formal proof can be found in the Appendix. The intuition for this regularity is best conveyed by the following argument. When the number of sellers increases, the impact of each individual seller on the expected gain that buyers obtain when visiting other sellers decreases. When the number of sellers is very large, this impact is very small, and if this effect is neglected entirely, the expected payoff of bidders who visit other sellers will be constant. A seller facing buyers with constant outside option optimally sets a zero entry fee (see e.g. Levin and Smith 1994). Ignoring this effect entirely is indeed at the core of the limiting equilibrium concepts proposed in McAfee (1993) and Peters and Severinov (1997).

4.2 Special cases: deterministic and uniformly distributed valuations

To illustrate how the equilibrium entry fee changes when the market grows in size, we focus on two special cases for the distribution of buyer valuations: deterministic valuations equal to 1, and uniformly distributed valuations over the interval [0, 1].

Corollary 3 (Deterministic Valuations). *Consider a market with* J *sellers and* I *buyers, and let the buyer cost of entering the market be zero. Let buyers have the same deterministic valuation of the goods, which is normalized to 1. Let sellers use second-price auctions, and let buyers play a symmetric equilibrium in which they bid their valuation. In this market, sellers charge an entry fee equal to*

$$\min\{E, R[1/J]\} = \min\left\{\frac{(I-1)(J-1)^{I-3}}{J^{I-1}}, \left(\frac{J-1}{J}\right)^{I-1}\right\}.$$

The proof is given in the Appendix. When buyer valuations are uniformly distributed, the equilibrium entry fee cannot be described in a closed form. Table 1 provides the entry fee depending on number of sellers[8] in markets with buyer to seller ratios $k = 1, 2, 3$ and 4. As can be observed, the entry fees decrease with the size of the market.

4.3 Buyer and seller surplus

Our model assigns different roles to buyers and sellers. Sellers seems to play a more active role as they act as mechanism designers. Yet, as we will see, this role does

[8] For buyer valuations uniformly distributed on [0, 1], it can easily be shown that $S_n = \frac{n-1}{n+1}$, $B_n = \frac{1}{n(n+1)}$ and $R[m] := \sum_{n=1}^{I} \Pr[n-1; m] \cdot B_n$.

Table 1 Equilibrium entry fee for uniformly distributed buyer values over the interval [0, 1]. The calculations are performed with Mathematica, and the numbers are rounded to the fourth decimal. The entry fees given in italics are equal to $R[1/J]$. In these markets, sellers expropriate the entire surplus from buyers

# sellers	Ratio of buyers to sellers $k = I/J$			
	$k = 1$	$k = 2$	$k = 3$	$k = 4$
1	*0.5000*	*0.1667*	*0.0833*	*0.0500*
2	0.1667	*0.1625*	*0.0893*	*0.0545*
3	0.0830	0.0920	0.0701	0.0497
4	0.0552	0.0592	0.0456	0.0328
5	0.0412	0.0435	0.0338	0.0244
6	0.0328	0.0343	0.0268	0.0195
7	0.0273	0.0284	0.0222	0.0162
8	0.0233	0.0241	0.0189	0.0138
9	0.0203	0.0210	0.0164	0.0121
10	0.0181	0.0186	0.0146	0.0107
100	0.0016	0.0016	0.0013	0.0009
1,000	0.0002	0.0002	0.0001	0.0001
$J \to \infty$	0	0	0	0

not seem to translate into a higher surplus. The next result provides a formula for the distribution of surplus among buyers and sellers in large markets for the two special cases of buyer valuations.

Corollary 4 *Consider a market with J sellers and $k \cdot J$ buyers, and let the buyer cost of entry into the market be zero. The table below reports the limit of the expected surplus of a buyer (BS) and a seller (SS) as $J \to \infty$ for deterministic buyer values (D) and for uniformly distributed buyer values (U).*

	BS	SS
D	$\frac{1}{e^k}$	$1 - \frac{k+1}{e^k}$
U	$\frac{1-\frac{(1+k)}{e^k}}{k^2}$	$\frac{2+k}{k \cdot e^k} + \frac{k-2}{k}$

The proof is given in the Appendix. It is easy to see that, in agreement with standard intuition, for both distributions the seller surplus increases in the ratio of buyers to sellers k and the buyer surplus diminishes in k. In particular, when $k \to \infty$, seller surplus converges to 1 and buyer surplus converges to 0. Conversely, when $k \to 0$, seller surplus converges to 0 and buyer surplus converges to 1. The case in which the market is populated by an equal number of buyers and sellers ($k = 1$) is particularly interesting because we can determine how the surplus is shared between one buyer and one seller. When valuations are uniformly distributed, a buyer obtains $\frac{e-2}{e} \approx 0.264$ and a seller earns $\frac{3-e}{e} \approx 0.104$. When buyers have the same valuation equal to one, the expected buyer surplus is $\frac{1}{e} \approx 0.368$ and the expected seller surplus is $\frac{e-2}{e} \approx 0.264$. Thus, in both cases, in large markets, buyers earn more than sellers although sellers design the trading mechanisms. It can easily be checked that in the case of uniformly distributed values, buyers obtain a larger share of the surplus (in percentage terms) compared to the case of deterministic values—an observation suggesting that buyers extract an informational rent when their valuations are private.

The case of equal number of buyers and sellers is also interesting because it allows a direct comparison to the literature on trading and efficiency in centralized markets. In a pure exchange economy with many commodities, Roberts and Postlewaite (1976) show that the gain an agent can achieve by acting monopolistically goes to zero with an increase in the size of the market.

Satterthwaite and Williams (1989) explore to what extent the strategic behavior of market participants holding private information may prevent markets from realizing all potential gains from trade. In their model, an equal number of buyers and sellers interact in a centralized market and trade occurs via a buyer's bid double auction—an auction in which both buyers and sellers submit bids, and the market price is determined by the highest price at which supply corresponds to demand. They show that, as the market becomes large, competitive pressures force market participants to bid close to their reservation values, leading to an efficient equilibrium outcome.

Satterthwaite and Shneyerov (2007) present an infinite horizon model with discounting and participation costs in which, in each period, a large number of buyers and sellers with privately observed values and costs are exogenously matched. Sellers hold first-price auctions and engage in trade if they find the highest bid satisfactory. The major conclusion of the analysis is that, as the time distance between the trading opportunities becomes small, the equilibrium prices converge to the Walrasian price, and the realized allocation converges to the competitive allocation.

Wooders (1998), on the other hand, analyzes a matching model in which, after a match occurs, a randomly chosen proposer makes a take-it-or-leave-it offer to his counterpart regarding the distribution of surplus. If an offer is rejected, the search process of both agents continues in the next round. The model differentiates between small and large markets, whereby in a small market the matching probabilities of a buyer and a seller depend on the decisions taken by the agents; in a large market, the matching probabilities are exogenously given. Wooders (1998) shows that, as the discount factor approaches one, the equilibrium distribution of surplus in a small market is not near the outcome of its corresponding large market because, in small markets, the surplus distribution depends on agents' behavior off the equilibrium path. Further, the equilibrium outcome is sensitive to the matching process—a finding that we will confirm for the present setting as well. As we will show, the distribution of surplus between buyers and sellers in the present model is quite different when buyers coordinate and when buyers randomize across sellers (see Sect. 7).

Abrams et al. (2000) analyze experimentally two polar imperfectly competitive market settings with random matching and search. Sellers post prices and buyers either purchase the goods at the posted price or conduct a costly search for another seller. In the "Diamond" treatment, each buyer is matched with one seller; in the "Bertrand" treatment, each buyer is matched with two sellers. Contrary to the theoretical prediction, according to which the entire surplus goes to sellers in the former model and to buyers in the latter model, the experimental results suggest a more equitable division of the surplus.

Gresik and Satterthwaite (1989) define the efficiency of a trading mechanism as the ratio of its ex ante expected gain from trade relative to the ex post efficient allocation. They analyze the optimal mechanism (i.e. individual rational and incentive compatible trading mechanism that maximizes the sum of buyers' and sellers' ex ante gains from trade) and find that the inefficiency decreases almost quadratically as the number of buyers and sellers increases. In the present model, the inefficiency arises because of the coordination problem inherent to this type of decentralized trading.

The trading inefficiency in the current setting can be determined with the measure proposed by Gresik and Sattertwaite. When the distribution of buyers and sellers is stochastic, the buyers with the J highest valuations may not be distributed across all sellers. Further, when entry is random and uncoordinated, items may even remain unsold. An efficient allocation is an allocation in which all the items are transferred to buyers. As $J \to \infty$, the surplus of one buyer and one seller is on average $1/e \approx 0.368$ while the efficient allocation provides an average surplus of $\frac{1}{2}$ (the expectation of the bidder valuation). We can conclude that $(\frac{1}{2} - 1/e)/\frac{1}{2} \approx 36\%$ of the surplus is lost due to the coordination problem resulting from the randomization strategies used by buyers.

5 Small markets

Here, we focus on markets with up to 2 sellers and up to 4 buyers. The equilibrium entry fees in a duopoly market are given as follows.

Corollary 5 *Consider a market game with 2 sellers and $n = 2, 3, 4$ buyers in which the sellers choose auctions with entry fees (or bonuses), and buyers have zero cost of entering the market. The entry fees in the equilibrium in which all buyers play a symmetric mixed strategy are given as follows.*

2 sellers and 2 buyers:

$$C^*_{2,2} = \frac{B_1 - B_2}{2}.$$

2 sellers and 3 buyers:

$$C^*_{2,3} = \min\left\{\frac{B_1 - B_3}{2}, \frac{B_1 + 2B_2 + B_3}{4}\right\}.$$

2 sellers and 4 buyers:

$$C^*_{2,4} = \min\left\{\frac{3 \cdot (B_1 + B_2 - B_3 - B_4)}{8}, \frac{B_1 + 3B_2 + 3B_3 + B_4}{8}\right\}.$$

These entry fees can easily be derived by a direct substitution into the formula given in Proposition 1 (the entry fee is $\min\{E, R[1/J]\}$). Therefore, the proof is omitted. For uniformly distributed buyer valuations, the equilibrium entry fees are reported in Table 2 and the surplus earned by a buyer and a seller in Table 3.

Table 2 Equilibrium entry fees in small markets

	1 buyer	2 buyers	3 buyers	4 buyers
1 seller	1/2	1/6	1/12	1/20
2 sellers	0	1/6	0.208	13/80

Buyer valuations are uniformly distributed. The calculations are performed with Mathematica and are available from the author upon request

Table 3 Seller surplus (left) and buyer surplus (right) in small markets

	1 buyer	2 buyers	3 buyers	4 buyers
1 seller	1/2, 0	2/3, 0	3/4, 0	4/5, 0
2 sellers	0, 1/2	1/4, 1/6	1/2, 0.021	0.614, 0

Buyer valuations are uniformly distributed

We observe that, when moving from a monopoly to a duopoly market, in general the equilibrium entry fee of a seller increases (an exception is the case of two buyers in which case the entry fee for a duopoly and a monopoly market is the same). While this observation might seem at first glance counterintuitive, it has a natural explanation. Observe that, in the case of two sellers, the number of bidders who visit each of the sellers is stochastic. This means that there is a certain chance that a seller will be visited by only one bidder, and in this case, the only source of revenue for the seller will be the entry fee. Due to the stochastic entry, a seller who competes with another seller earns less by holding the auction (compared to the monopoly case) and leaves a higher surplus to buyers in his auction. Therefore, a seller has an incentive to increase his entry fee when he faces competition from another seller.

To further understand how the size of the market affects the surplus of buyers and sellers, it is useful to focus on the trade-offs associated with market replication. As the market grows, there are two effects that work in opposite directions. On the one hand, as we argued earlier, the greater the number of buyers and sellers present in the market, the smaller will be the effect of a seller deviation on the buyer expected payoffs with other sellers. Because of this feature of the market setting, the competition among sellers intensifies with the size of the market, and sellers lower their entry fees in equilibrium as the market becomes larger. On the other hand, as the market size increases, it becomes less likely that a buyer will have no competitors when visiting one of the sellers.[9] In that sense, the competition in the auction of each individual seller intensifies as well. Whether buyers obtain a positive surplus depends on the magnitudes of these two effects. In particular, if the buyer expected surplus in the auction $R[1/J]$ exceeds the entry fee E, buyers obtain a positive surplus. Since E converges to zero when the market becomes large, buyers will start earning a positive surplus once the market has reached a certain size. The point at which this happens depends on the distribution of buyer valuations, and the ratio of buyers to sellers k.

[9] The probability that a buyer will have no competitors equals $(1 - \frac{1}{J})^{kJ-1}$, which is decreasing in J.

Table 3 illustrates that in markets with 1 seller and 2 buyers and 2 sellers and 4 buyers, buyers realize a surplus of zero. Only when the market grows and reaches the size of 3 sellers and 6 buyers (or more participants), do buyers earn a positive surplus in equilibrium. A comparison of the magnitude of these two opposite effects can also explain the earnings of sellers. When the size of the market grows, but the ratio of buyers and sellers stays the same, the seller surplus declines because the extra revenue earned in the auctions cannot compensate for the lower revenues earned from entry fees. Indeed, in the case of duopoly with four buyers, the entry fees are lower compared to the monopoly case with two buyers. A seller would prefer to have two buyers for sure (i.e. to be a monopolist) rather than to share four buyers with another seller and have each buyer visit him with a probability of $\frac{1}{2}$. Finally, when one more bidder enters the market, holding the number of sellers constant, the surplus of each seller increases. The increased earnings of sellers in the auctions outweigh the decreased proceeds from entry fees. This feature conforms to our standard economic intuition.

6 Variable entry fees

So far, we discussed the equilibria in which sellers use entry fees that do not depend on the number of buyers. While the restriction to this type of equilibria seems natural, there might be other equilibria that lead to different division of the surplus between buyers and sellers. In this subsection, we will explore this possibility by describing all equilibria in a market with 2 buyers and 2 sellers in which sellers use variable entry fees. Our main conclusion is given in the next proposition.

Proposition 2 *Consider a market with two sellers and two buyers. Let the buyer cost of entry into the market be zero and let sellers be allowed to use variable entry fees. The entry fees C^{*1} in the case of one bidder and C^{*2} in the case of two bidders given by*

$$C^{*1} = E = (B_1 - B_2)/2,$$
$$B_2 - B_1 \leq C^{*2} \leq (B_1 + 3B_2)/2$$

constitute all symmetric equilibrium entry fees. Buyers enter the market with a probability of one. In all equilibria, total surplus equals $B_1 + B_2$. Any division of the total surplus between buyers and sellers can arise as an equilibrium outcome. Buyers earn the entire surplus when $C^{*2} = B_2 - B_1$ and sellers expropriate the entire surplus when $C^{*2} = (B_1 + 3B_2)/2$.

The proof is provided in the Appendix. The multiplicity of equilibria under variable entry fees suggests that sellers can use the entry fees as a collusive devise. By charging higher entry fees when there is competition in the auction (i.e. when there are two bidders), sellers can increase their surplus and lower the buyer surplus down to zero. As the proposition demonstrates, such a seller behavior can be an equilibrium outcome.

7 Buyer coordination equilibria

Proposition 1 described a class of equilibria in which all buyers symmetrically randomize across sellers. In this section, we discuss equilibria in which buyers coordinate—they either choose a particular seller or stay out of the market with a probability of one. For expositional clarity and in order to facilitate a comparison with related models (i.e. Moldovanu et al. 2008), we will consider the case of 2 sellers. A pure strategy of buyer i, $\sigma_i(C_1, C_2) : \mathbb{R} \times \mathbb{R} \rightarrow \{0, 1, 2\}$ specifies the deterministic location decision of each buyer, given the entry fees chosen by the sellers.

Proposition 3 *Consider a market game with $J = 2$ sellers in which the sellers choose auctions with an entry fee (or bonus) and buyers play pure continuation strategies. Let the number of buyers be either even ($I = 2 \cdot n$) or odd ($I = 2 \cdot n + 1$). The equilibria in this market are given as follows:*

- *If $B_{n+1} > c$ the game has no equilibrium.*
- *If $B_{n+1} \leq c$ two possibilities exist. Let $l \in \{0, 1, 2 \ldots, n\}$ be the maximum number of bidders who can enter an auction (with no entry fee) and still realize a surplus higher than their cost of entry: $B_{l+1} \leq c < B_l$.*

Case 1: *$B_{l+1} < c$. The game has a unique equilibrium which is symmetric. Each seller charges an entry fee $C = B_l - c$ and is visited by a number of l buyers. The remaining buyers (if there are any) stay out of the market. Sellers expropriate the entire surplus from buyers.*

Case 2: *$B_{l+1} = c$. The game has four equilibria in which each seller charges either an entry fee of $B_l - c$ and is visited by l buyers or charges an entry fee of $B_{l+1} - c$ and is visited by $l + 1$ buyers. All equilibria are expected payoff equivalent. In all equilibria, the sellers expropriate the entire surplus from buyers (i.e. leave each buyer a surplus equal to his cost of entry c). The profit of each seller is the same and equals $S_l + l \cdot B_l - l \cdot c$ in all equilibria.*

The proof is given in the Appendix. The proposition implies that in a market with sufficiently many buyers and a sufficiently high cost of entry into the market, the game has an equilibrium in which buyers play pure strategies. When the cost of entry is low or when the number of buyers is small, the game has no equilibrium. Moldovanu et al. (2008) consider a similar model in which two sellers decide how many units to sell in a uniform price auction and buyers coordinate. In line with the result presented here, they find that, when the seller marginal cost is sufficiently small, there is no equilibrium in which both sellers are active and make positive profits. In our setting, the choice variable for sellers is the entry fee rather than the number of units produced, yet the structure of the problem and the result are similar.

Bulow and Klemperer (1996) analyze the problem of a monopolist who auctions off an item to a fixed number of bidders. They find that attracting an additional bidder to participate in an auction with no reserve price is more valuable for the seller than any mechanism designed to extract as much surplus as possible from the existing bidders (i.e. an auction with optimally set reserve price). Similar intuition applies to our setting as well. When the cost of entering the market is small, each seller prefers to lower his cost of entry so as to poach one more buyer from his competitor. The intuition gained from the monopoly model breaks down, however, once bidders are allowed to play randomization strategies. In Bulow and Klemperer's model, buyers are locked into the mechanism of the seller, and their payoff decreases once a new bidder enters the auction. In the present model, in which entry is endogenous, bidders will react to the entry of one additional bidder by reducing their entry probability and increasing the probability with which they go to other sellers or stay out of the market.

8 Discussion

This paper presents an analytically tractable model of decentralized trading in which sellers compete for buyers by offering transaction mechanisms. The model analyzes markets with a finite number of buyer and sellers, and the interaction among market participants is purely strategic. In all equilibria, sellers hold auctions with a trivial reserve price (equal to sellers' use value of the goods) and an entry fee. In general, the entry fee can depend on the number of buyers who attend the auction. When the entry fees can vary depending on the number of buyers, multiple equilibrium outcomes are possible. In a market for two buyers and two sellers only, we showed that any distribution of the surplus can be sustained as an equilibrium outcome.

There is a unique equilibrium, in which sellers do not vary their entry fees with the number of participants in their auctions, and buyers randomize symmetrically across sellers. We provide a formula for the equilibrium entry fee, which depends on the number of buyers and sellers, the distribution of buyer values, and the buyer cost of entering into the market. In this equilibrium, buyers in general can earn a surplus higher than their cost of entry. Equilibria are also studied in which buyers coordinate across sellers. These equilibria exist when the cost of entry is sufficiently high and the number of bidders is sufficiently large so that each seller finds it too costly to deviate and attract an additional buyer. In all these equilibria, buyers earn a surplus just enough to cover their cost of entering the market.

The model we analyzed here is static in nature. Once buyers stochastically distribute across sellers, and the trading occurs, the game ends. It can be expected that buyers and sellers who were unable to trade can attempt to trade at a later point, and such a contingency will have an effect on the sellers' mechanism design problem. If buyers and sellers interact repeatedly, the trading process will be influenced by signaling, learning and other dynamic effects. Such effects have recently been explored, both theoretically and experimentally, in simple monopoly settings. Salmon and Wilson (2008) consider the problem of a monopolist who offers a unit in an auction and uses the information revealed in the bidding process to make a take-it-or-leave-it price offer to one of the unsuccessful bidders. They show experimentally that this mechanism generates higher revenue for the seller than holding two sequential auctions. Zeithammer (2009) analyzes a related model in which the monopolist can hold a second auction only when demand revealed in the first auction is strong enough to cover his opportunity cost for the second item. In anticipation of the second auction, bidders strategically lower their bids in the first auction, and Zeithammer (2009) analyzes various forms of commitment for the seller not to offer the second item for sale. Chade and de Serio

(2002) study the problem of a monopolist who does not know the quality of the item he puts for sale and updates his value as he randomly meets buyers. The authors show that, in this strategic setting, the seller's incentive to experiment longer by keeping the price of the item high is not necessarily increasing in the precision of the signals the seller receives from buyers.

Models in which sellers compete by designing mechanisms and interact with buyers repeatedly can reveal interesting signaling and learning effects that are missing in the present framework. Understanding this type of dynamic market interaction presents an exciting and a challenging topic for future research.

Appendix

Proof of Proposition 1. The proof proceeds in three steps. First, we describe how buyers distribute across sellers depending on the entry fee C_j of seller j and C of all other sellers. Then, we formulate the expected payoff of seller j in the first stage of the game as a function of the entry fees by taking into account the distribution of buyers across sellers in the continuation equilibrium. Finally, we show that, if all other sellers choose their entry fees as specified in the proposition, it is a best response for seller j to use the same entry fee. □

Buyer continuation equilibria (second stage)

The following three scenarios present all possibilities for the distribution of buyers across sellers depending on the entry fees C_j and C.

Scenario A. *Buyers enter the market with a probability of one and visit all the sellers.* The probability m with which buyers visit seller j is determined by the equation

$$R[m] - C_j = R\left[\frac{1-m}{J-1}\right] - C. \tag{A}$$

This equation has a solution $m \in [0, 1]$, and for this solution, the inequality $R[m] - C_j \geq c$ holds.

Scenario B. *Buyers enter the market with a probability smaller than one and visit all the sellers.* Equation (A) has a solution $m \in [0, 1]$, and for this solution, $R[m] - C_j < c$. Buyers visit seller j with a probability determined by the equation

$$R[m] - C_j = c, \tag{B}$$

and this equation has a solution in the interval $[0, 1]$.

Scenario C. *Buyers either strictly prefer to visit only seller j or they strictly prefer to distribute only across the other sellers.* The entry fees C_j and C are such that equation (A) is not satisfied for any $m \in [0, 1]$. It can be easily verified that in this scenario C_j is not a best response to C as for the C_j for which this scenario applies seller j either leaves too much surplus to buyers or excessively discourages entry.

The constellations of C_j and C, which give rise to the three scenarios, are summarized in Fig. 2.

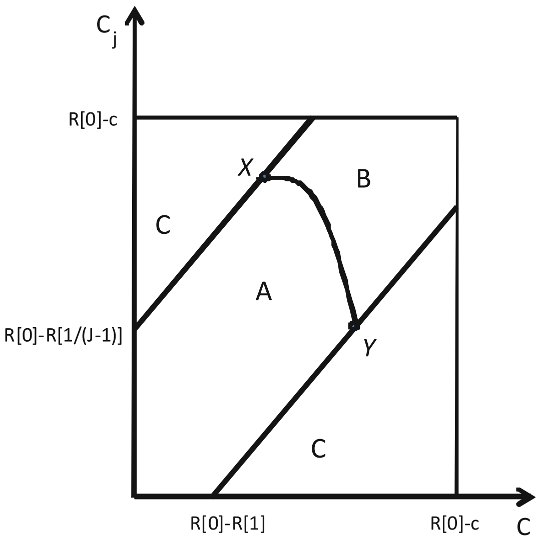

Fig. 2 Areas of entry fees C and C_j for which the distribution of buyers corresponds to scenarios A, B, and C. Along the curve $X - Y$ buyers enter the market with a probability of one and earn a surplus equal to their cost of entry (i.e. the entry probability m satisfies both Eq. A and B)

The following monotonicity property ensures that, for all C_j and C, there is a unique symmetric mixed strategy equilibrium for buyers in each of the three scenarios.

Lemma 4 *$R[m]$ is strictly decreasing in m.*

Proof Assume that all competitors of bidder i participate in the auction with a probability of m, except for one of them, named k, who participates with a probability of m_k. The payoff of bidder i is

$$R[m_k, m] = \sum_{n=0}^{I-2} \Pr[n; m] \cdot (m_k \cdot B_{n+1} + (1 - m_k)B_n).$$

The derivative with respect to the entry probability of all other bidders is

$$\frac{\mathrm{d}R[m]}{\mathrm{d}m} = (I-1) \cdot \left.\frac{\partial R[m_k, m]}{\partial m_k}\right|_{m_k = m} = (I-1) \cdot \sum_{n=0}^{I-2} \Pr[n; m](B_{n+1} - B_n) < 0.$$

The derivative is negative because $B_{n+1} < B_n$ for all n. □

Competition among sellers (first stage) The expected profit of seller j is given by

$$\Pi_j(C_j, C) = \sum_{n=1}^{I} \Pr[n; m] \cdot S_n + I \cdot m \cdot C_j,$$

where m is the probability with which buyers go to seller j given the entry fees C_j and C. When seller j chooses the entry fee C_j, seller j indirectly determines the entry probability m. Therefore, the decision of seller j can alternatively be viewed as a choice of the entry probability m, whereby the entry fee C_j is indirectly determined from Eq. (A) or Eq. (B). We solve Eqs. (A) and (B) for C_j and express the payoff of seller j as a function of m and C as follows:

$$\Pi_j(m,C) = \begin{cases} \sum_{n=1}^{I} \Pr[n;m]\cdot S_n + I\cdot m\cdot (R[m] - R[\frac{1-m}{J-1}] + C) & \text{for Eq. (A)},\\ \sum_{n=1}^{I} \Pr[n;m]\cdot S_n + I\cdot m\cdot (R[m] - c) & \text{for Eq. (B)}.\end{cases}$$

The expected payoff of seller j can be alternatively expressed as

$$\Pi_j(m,C) = \begin{cases} JS(m) - I\cdot m\cdot (R[\frac{1-m}{J-1}] - C) & \text{if Eq. (A) holds},\\ JS(m) - I\cdot m\cdot c & \text{if Eq. (B) holds},\end{cases}$$

where $JS(m) = \sum_{n=1}^{I} \Pr[n;m]\cdot S_n + I\cdot m\cdot R[m]$ is the expected joint surplus of one seller and all buyers, given that each buyer visits this seller with a probability of m. The following relationship will be useful for the subsequent equilibrium analysis.

Lemma 5 *Assume all bidders visit a seller with a probability of m. The marginal change of $JS(m)$ equals the sum of all bidders' gains:*

$$JS'(m) = I\cdot R[m].$$

Proof Let $JS(m_i, m)$ denote the joint surplus of the seller and all bidders, if bidder i participates with a probability of m_i and all other bidders with a probability of m. The next argument draws on the following property of the independent private value model. □

Lemma 6 (Levin and Smith 1994) *When one bidder joins an auction in which $(n-1)$ bidders are participating (with a probability of one), the change in the Joint Surplus is equal to the individual bidder's gain:*

$$JS_n - JS_{n-1} = B_n.$$

A concise proof of this relationship can be found in (Levin and Smith, 1994, p. 592). The expected gain of a bidder who participates in an auction with a probability of m_i, given that all other bidders participate with a probability of m is $m_i\cdot R[m]$, and therefore, using the above lemma, we obtain $JS(m_i, m) = m_i\cdot R[m]$. The identities

$$JS'(m) = I\cdot \frac{\partial}{\partial m_i}(JS(m_i,m))\Big|_{m_i = m} = I\cdot R[m]$$

yield the desired result. □

Case 1. Let all other sellers charge E. We proceed in two steps. In step 1, we show that $\Pi_j(E,E) > \Pi_j(C_j,C)$ for all $C_j \le R[1] - R[0] + E$. In step 2, we show that $\Pi_j(E,E) > \Pi_j(C_j,C)$ for all $C_j > R[1] - R[0] + E$.

Step 1. In this case, buyers distribute according to Eq. (A). Showing that $\Pi_j(C_j, E)$ is maximized for $C_j = E$ is equivalent to showing that $\Pi_j(m, E)$ is maximized for $m = 1/J$. That is, we need to show that the first-order condition holds for $m = 1/J$. Using Lemma 5, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}m}\Pi_j\,(m, C) = I \cdot R[m] - I \cdot \frac{\mathrm{d}}{\mathrm{d}m}\left[m \cdot \left(R\left[\frac{1-m}{J-1}\right] - E\right)\right] = 0$$
$$\Leftrightarrow I \cdot \left[R\,[m] - \left(R\left[\frac{1-m}{J-1}\right] - E\right) - m \cdot \frac{\mathrm{d}}{\mathrm{d}m}\left(R\left(\frac{1-m}{J-1}\right)\right)\right] = 0$$
$$\Leftrightarrow E = \left[m \cdot \frac{\mathrm{d}}{\mathrm{d}m}\left(R\left(\frac{1-m}{J-1}\right)\right)\right]\Bigg|_{m\,=\,1/J}.$$

This equation obviously holds true as it corresponds to the definition of E.

Step 2. In this case, buyers distribute according to Eq. (B). The partial derivative of the expected payoff of seller j with respect to m equals

$$\frac{\mathrm{d}}{\mathrm{d}m}\Pi_j(m, E) = I \cdot (R[m] - c).$$

$\Pi_j(m, E)$ reaches its maximum when $R[m] - c = 0$, and this result combined with Eq. (B) yields that $\Pi_j(C_j, E)$ reaches its maximum for $C_j = 0$ and is decreasing in C_j for all $C_j \geq 0$.

Case 2. Let all other sellers charge $R[1/J] - c$. When $C_j \leq R[1/J] - c$ buyers distribute according to Eq. (A) and when $C_j \leq R[1/J] - c$, they distribute according to Eq. (B). In the former case, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}m}\Pi_j[m, (R[1/J] - c)] = I \cdot \left[R[m] - R\left[\frac{1-m}{J-1}\right] + (R[1/J] - c)\right.$$
$$\left. -m \cdot \frac{\mathrm{d}}{\mathrm{d}m}(R\left(\frac{1-m}{J-1}\right)\right].$$

Observe that $\frac{\mathrm{d}}{\mathrm{d}m}\Pi_j[m, (R[1/J] - c)]\big|_{m=1/J} = I \cdot (R[1/J] - c - E) < 0$ which implies that the expected payoff of seller j decreases with an increase in m beyond $1/J$. In other words, lowering C_j below $R[1/J] - c$ leads to a lower expected payoff for seller j. In the latter case, we know that $\Pi_j[C_j, (R[1/J] - c)]$ decreases in C_j for $C_j \geq 0$ (see the analysis of Case 1), so increasing C_j above $R[1/J] - c$ is not optimal for seller j either.

Case 3. When $c \geq R[1/J]$ and all sellers charge an entry fee of zero, bidders enter the market with a probability less than one. The entry probability is determined by Eq. (B), and, as already argued earlier, $C_j = 0$ maximizes the expected payoff of seller j. □

Proof of Corollary 2. Let $R[m_i, m]$ be the expected payoff of a bidder who participates in an auction with no entry fee or a reserve price, given that all other bidders participate with a probability of m except for one, who enters the auction with a probability of m_i. Then,

$$\begin{aligned}
\left.\frac{\mathrm{d}R[m]}{\mathrm{d}m}\right|_{m=1/J} &= I \cdot \left.\frac{\partial}{\partial m_i}R[m_i, m]\right|_{m_i=m=1/J} \\
\Leftrightarrow \left.\frac{\mathrm{d}R[m]}{\mathrm{d}m}\right|_{m=1/J} &= kJ \cdot \left.\frac{\partial}{\partial m_i}(m_i \cdot R[1,m] + (1-m_i)\cdot R[0,m])\right|_{m_i=m=1/J} \\
\Leftrightarrow \left.\frac{\mathrm{d}R[m]}{\mathrm{d}m}\right|_{m=1/J} &= kJ \cdot \left.(R[1,m] - R[0,m])\right|_{m=1/J}.
\end{aligned}$$

Since $E_{J,kJ} = -\frac{1}{J\cdot(J-1)} \cdot \frac{\mathrm{d}R[m]}{\mathrm{d}m}\big|_{m=1/J}$, we obtain $E_{J,kJ} = \frac{k}{J-1} \cdot (R[0,m] - R[1,m])\big|_{m=1/J}$. Recall that $R[m] = \sum_{n=1}^{I} \Pr[n-1; m] \cdot B_n$ and observe that

$$R[0,m] - R[1,m] = \sum_{n=1}^{kJ} \Pr[n-1; m] \cdot (B_{n-1} - B_n) < \sum_{n=1}^{kJ} \Pr[n-1; m] \cdot B_0 = B_0.$$

The expression $R[0,m] - R[1,m]$ is bounded from above for all m, and therefore $E_{J,kJ}$ converges to zero as $J \to \infty$. □

Proof of Corollary 3. As all buyers have a valuation of 1, and bid their valuation, we have $B_1 = 1$, and $B_n = 0$ for all $n > 1$. Hence,

$$R[1/J] = \left(\frac{J-1}{J}\right)^{I-1}$$

and

$$E = \frac{(I-1)(J-1)^{I-3}}{J^{I-1}}.$$

Applying Proposition 1 (see Cases 1&2), we obtain the desired result. □

Proof of Corollary 4. Deterministic valuations: From Proposition 2 we know that the entry fee converges to zero. Therefore, the surplus of a buyer converges to the probability with which a buyer will be the only customer who visits a certain seller, which is

$$\lim_{J\to\infty}\left(1-\frac{1}{J}\right)^{kJ-1} = \lim_{J\to\infty}\left[\left(1-\frac{1}{J}\right)^{J}\right]^{k} = \frac{1}{e^k}.$$

The total surplus generated from trade equals the probability with which each seller will sell his item multiplied by the number of sellers, which is

$$J \cdot \lim_{J\to\infty}\left(1-\left(1-\frac{1}{J}\right)^{kJ}\right) = J \cdot \left(1-\frac{1}{e^k}\right).$$

The surplus of one seller is then

$$\frac{J \cdot (1 - \frac{1}{e^k}) - kJ \cdot \frac{1}{e^k}}{J} = 1 - \frac{k+1}{e^k}.$$

□

Uniformly distributed valuations: The seller surplus is

$$\lim_{J\to\infty} \sum_{n=1}^{k\cdot J} \Pr[n;\, 1/J] \cdot S_n = \lim_{J\to\infty} \sum_{n=1}^{k\cdot J} \Pr[n;\, 1/J] \cdot \frac{n-1}{n+1}$$

$$= 1 - \lim_{J\to\infty} \left(1 - \frac{1}{J}\right)^{kJ} - 2 \cdot \lim_{J\to\infty} \sum_{n=1}^{k\cdot J} \Pr[n;\, 1/J] \left(\frac{1}{n+1}\right)$$

$$= \left(1 - \frac{1}{e^k}\right) - 2\frac{\left(1 - \frac{k+1}{e^k}\right)}{k} = \frac{2+k}{k \cdot e^k} + \frac{k-2}{k}.$$

The buyer surplus is

$$\lim_{J\to\infty} \sum_{n=1}^{k\cdot J} R[1/J] = \lim_{J\to\infty} \sum_{n=1}^{k\cdot J} \Pr[n-1;\, 1/J] \cdot B_n = \lim_{J\to\infty} \sum_{n=1}^{k\cdot J} \Pr[n-1;\, 1/J] \cdot \frac{1}{n(n+1)}$$

$$= \lim_{J\to\infty} \sum_{n=1}^{k\cdot J} \Pr[n-1;\, 1/J] \cdot \frac{1}{n} - \lim_{J\to\infty} \sum_{n=1}^{k\cdot J} \Pr[n-1;\, 1/J] \cdot \frac{1}{n+1}$$

$$= \frac{k - \frac{k}{e^k}}{k^2} - \frac{k - 1 + \frac{1}{e^k}}{k^2} = \frac{1 - \frac{(1+k)}{e^k}}{k^2}.$$

□

Proof of Proposition 2. The proof is organized in two lemmas. First, we show that it is not optimal of any of the sellers to use such entry fees that lead to buyers staying out of the market with a positive probability (Lemma 7). This statement allows us to focus only on the cases in which bidders randomize between the two sellers. Then, we show that the entry fees described in the proposition constitute all equilibria (Lemma 8).

□

Lemma 7 *Let one of the sellers (say, seller 2) charge the entry fees $\tilde{C}^1$ and $\tilde{C}^2$, and let the other seller (seller 1) charge the entry fees C^1 and C^2. Let these entry fees be such that in the continuation equilibrium buyers go to seller 1 with a probability of m, stay out of the market with a probability of $m_0 > 0$, and go to seller 2 with a probability of $1 - m - m_0$. Then, it is a profitable deviation for seller 1 to lower the entry fees so as to allow bidders to enter in his auction with a probability of $m + m_0$.*

Proof If the seller charges a non-variable entry fee of $C = m \cdot C^1 + (1 - m) \cdot C^2$ buyers will continue to visit this seller with a probability of m, and the expected payoff of seller 1 will be the same as with the entry fees C^1 and C^2. If we assume that bidders stay out of the market with positive probability, their entry is given by Eq. (B) and the expected payoff of seller 1 is $\Pi_1(m, C) = JS(m) - I \cdot m \cdot c = JS(m)$. Lemma 4 asserts that $JS'(m) = I \cdot R[m] > 0$, so the expected payoff of seller 1 increases when seller 1 lowers his entry fee to allow all bidders to enter the market with a probability of one. □

Lemma 8 *The entry fees described in the proposition constitute all equilibrium entry fees.*

Proof From the previous lemma, we know that in any equilibrium, the distribution of buyers is given by Eq. (A). The non-variable entry fee $C = m \cdot C^1 + (1-m) \cdot C^2$ leads to the same distribution of buyers as the entry fees C^1 and C^2, and from equation (A), we obtain

$$C = m \cdot (2B_2 - 2B_1 + \tilde{C}^1 - \tilde{C}^2) + (B_1 - B_2 + \tilde{C}^2).$$

The expected payoff of seller 1 can be expressed as

$$\Pi_1(m, C; \tilde{C}^1, \tilde{C}^2) = m^2(S_2 + 2 \cdot (2B_2 - 2B_1 + \tilde{C}^1 - \tilde{C}^2)) + 2m(B_1 - B_2 + \tilde{C}^2).$$

The first-order condition yields

$$\frac{\mathrm{d}}{\mathrm{d}m} \Pi_j(m, ; \tilde{C}^1, \tilde{C}^2)\bigg|_{m\,=\,1/2} = 0 \Leftrightarrow C^{*1} = B_1 - B_2 - S_2/2.$$

From Lemma 6 follows[10] that $S_2 = (B_1 - B_2)$, and we obtain $C^{*1} = (B_1 - B_2)/2$. It can be easily checked that the second-order condition is satisfied for the range of entry fees given in the proposition and that the entry fees $C^{*1} = (B_1 - B_2)/2$ and $B_2 - B_1 \leq C^{*2} \leq (B_1 + 3B_2)/2$ do not lead to a negative expected payoff for buyers and sellers. Hence, the entry fees described in the proposition constitute all equilibria, and any division of the total surplus of $B_1 + B_2$ arises as an equilibrium outcome. □

Proof of Proposition 3. The proof is organized in several steps. The next three lemmas establish some useful equilibrium properties (should equilibria exist). Lemma 9 states that in equilibrium, the entry fees are such that buyers do not earn a positive surplus net of their cost of entry. Knowing this property, in Lemma 10 we establish that in any equilibrium, the number of buyers who visit the two sellers cannot differ by more than one. And in Lemma 11, we show that, if $B_{l+1} < c$, the number of buyers in equilibrium must be the same. So, the number of bidders in equilibrium can differ

[10] According to Lemma 6

$$\begin{aligned} n \cdot B_n + S_n - [(n-1) \cdot B_{n-1} + S_{n-1}] &= B_n \\ \Leftrightarrow (n-1) \cdot (B_n - B_{n-1}) &= S_n - S_{n-1}, \end{aligned}$$

and for $n = 2$, we obtain $S_2 = B_1 - B_2$.

only if $B_{l+1} = c$. Using these results, we conclude that if equilibria exist, then these equilibria can only be the ones described in the proposition. As a final step, we verify that for the strategy profiles specified in the proposition, sellers do not have profitable deviations. □

Lemma 9 *In all equilibria (if such exist) each seller expropriates the entire surplus from buyers that visit him.*

Proof Assume by contradiction that there are equilibria in which buyers realize a surplus higher than c when going to (at least) one of the sellers. Let sellers choose the entry fees C_1 and C_2 and let according to the continuation equilibrium a number of n_1 buyers go to seller 1 and n_2 buyers go to seller 2. Without loss of generality, we can assume that seller 1 gives a surplus to buyers not lower than seller 2. Two cases for the payoffs of buyers are possible:

Case 1: $B_{n_1} - C_1 > B_{n_2} - C_2 = c,$
Case 2: $B_{n_1} - C_1 \geq B_{n_2} - C_2 > c.$

In Case 1, seller 1 obviously can marginally increase his entry fee without losing a customer. This applies also for Case 2 when $B_{n_1} - C_1 > B_{n_2} - C_2$. The more interesting case is $B_{n_1} - C_1 = B_{n_2} - C_2 > c$. By increasing his entry fee marginally, seller 1 will lose a customer if this customer either exits the market or switches to seller 2. If the increase in C_1 is sufficiently small, exiting the market is not profitable. If a customer goes to seller 2, this customer will earn $B_{n_2+1} - C_2$, a payoff which is strictly lower than $B_{n_2} - C_2$ (recall that B_n is decreasing in n) and thus lower than $B_{n_1} - C_1$. So, a seller can increase his entry fee marginally without losing a customer if not all bidders who enter the market earn exactly c. □

Lemma 10 *In all equilibria (if such exist), the number of bidders going to the two sellers cannot differ by more than one bidder.*

Proof Assume by contradiction that $n_1 > n_2 + 1$. The profit of each seller equals the joint surplus of the buyers and the seller minus the surplus of the buyers. Note that, as we established in Lemma 9, in equilibrium each buyers earns on average c from participating in the market. Thus, the profit of a seller who is visited by n bidders is $JS_n - n \cdot c$. From Lemma 6 (Levin and Smith 1994), we know that the reduction in the joint surplus when one bidder leaves the auction equals the expected payoff of this bidder: $JS_n - JS_{n-1} = B_n$. In order for seller 1 not to find it profitable to increase his entry fee and lose one bidder, it must be that the reduction in profit associated with losing this bidder, which equals B_{n_1}, be higher than the surplus which the seller needs to provide to this bidder, which equals c. Similarly, in order that seller 2 does not find it attractive to gain another customer, the increase in the total surplus when one customer joins, $B_{n_{2+1}}$, should be lower than the cost c. Combining these two requirements, we obtain $B_{n_1} \geq c \geq B_{n_{2+1}}$. But on the other hand, since $n_2 + 1 < n_1$, we have $B_{n_1} < B_{n_2+1}$, a contradiction. □

Lemma 11 *If $B_{l+1} < c$ no equilibrium exists in which the sellers use different entry fees and are visited by a different number of buyers.*

Proof Using the same arguments as in the previous lemma, we obtain $B_{n_1} > c > B_{n_2+1}$. But since $n_2 + 1 \leq n_1$, we have $B_{n_2+1} \leq B_{n_1}$, a contradiction. □

Next, we prove that if $B_{n+1} > c$, the game has no equilibrium. From the three lemmas, we know that in equilibrium (if one exists), one of the sellers (say, seller 1) has n bidders and charges an entry fee of $B_n - c$ and the other seller has either n or $n + 1$ bidders and charges an entry fee of either $B_n - c$ or $B_{n+1} - c$, respectively. Seller 1 can attract one more bidder by giving this bidder an expected payoff of c. By attracting this additional bidder, seller 1 will increase the joint surplus by B_{n+1}. That is, seller 1 will be able earn additionally $B_{n+1} - c$ by reducing the entry fee and allowing one more bidder to joint his auction. When $B_{n+1} > c$ poaching one bidder from the other seller is a profitable deviation for seller 1. Hence, there are no equilibria when $B_{n+1} > c$.

To prove that the entry fees and the buyer continuation strategies stated in the proposition are equilibria, we will demonstrate that no profitable deviation of a seller exists. Consider first the case in which a seller who has l bidders increases his entry fee and loses one customer. The gain of such an increase in the entry fee equals the amount this bidder earns from participation, c, and the loss equals this bidder's contribution to the joint surplus, which is B_l. Since $B_l < c$ such a deviation is not profitable. Consider now the case in which the seller who has l bidders lowers his entry fee to poach one more customer. When $B_{l+1} < c$, lowering the entry fee is also not profitable. Thus, the symmetric profile described in the proposition is equilibrium. When $B_{l+1} = c$, then a seller is indifferent between having $l + 1$ or l bidders, and therefore there are four equilibria with identical payoffs as described in the proposition. □

References

Abrams, E., Sefton, M., Yavas, A.: An experimental comparison of two search models. Econ Theor **16**, 735–749 (2000)

Bernheim, D., Whinston, M.: Menu actions, resource allocation, and economic influence. Quart J Econ **101**, 1–32 (1986)

Bulow, J., Klemperer, P.: Auctions versus negotiations. Am Econ Rev **86**, 180–194 (1996)

Burguet, R., Sakovics, J.: Imperfect competition in auction designs. Int Econ Rev **40**, 231–247 (1999)

Chade, H., deSerio, V.V.: Pricing, learning, and strategic behavior in a single-sale model. Econ Theor **19**, 333–353 (2002)

Chakraborty, I., Kosmopoulou, G.: Auctions with endogenous entry. Econ Lett **72**, 195–200 (2001)

Eeckhout, J., Kircher, P.: Sorting versus screening: search frictions and competing mechanisms. J Econ Theor **117**, 861–913 (2010)

Engelbrecht-Wiggans, R.: Optimal auctions revisited. Games Econ Behav **5**, 227–239 (1993)

Epstein, L., Peters, M.: A revelation principle for competing mechanisms. J Econ Theor **88**, 119–161 (1999)

Gresik, T.A., Satterthwaite, M.A.: The rate at which a simple market converges to efficiency as the number of traders increases: an asymptotic result for optimal trading mechanisms. J Econ Theor **48**, 304–332 (1989)

Levin, D., Smith, J.L.: Equilibrium in auctions with entry. Am Econ Rev **84**, 585–599 (1994)

Lu, J.: Optimal entry in auctions with valuation discovery costs. Appl Econ Res Bull **2**, 22–30 (2008)

Lu, J.: Auction design with opportunity cost. Econ Theor **38**, 73–103 (2009)

Marmer, V., Shneyerov, A., Xu, P.: What model of entry in first-price auctions? A nonparametric approach. Working paper, University of British Columbia (2007)

McAfee, R.P.: Mechanism design by competing sellers. Econometrica **61**, 1281–1312 (1993)

McAfee, R.P., McMillan, J.: Auctions with entry. Econ Lett **23**, 343–347 (1987)

Moldovanu, B., Sela, A., Shi, X.: Competing auctions with endogenous quantities. J Econ Theor **141**, 1–27 (2008)
Peck, J.: Competition in transaction mechanisms: The emergence of price competition. Games Econ Behav **16**, 109–123 (1996)
Peters, M.: A competitive distribution of auctions. Rev Econ Stud **64**, 97–123 (1997)
Peters, M.: Common agency and the revelation principle. Econometrica **69**, 1349–1372 (2001)
Peters, M., Severinov, S.: Competition among sellers who offer auctions instead of prices. J Econ Theor **75**, 141–179 (1997)
Roberts, D.J., Postlewaite, A.: The incentives for price-taking behavior in large exchange economies. Econometrica **44**, 115–127 (1976)
Salmon, T.C., Wilson, B.J.: Second chance offers versus sequential auctions: Theory and behavior. Econ Theor **34**, 47–67 (2008)
Samuelson, W.: Competitive bidding with entry cost. Econ Lett **17**, 53–57 (1985)
Satterthwaite, M., Shneyerov, A.: Dynamic matching, two-sided incomplete information, and participation costs: Existence and convergence to perfect competition. Econometrica **75**, 155–200 (2007)
Satterthwaite, M.A., Williams, S.R.: The rate of convergence to efficiency in the buyer's bid double auction as the market becomes large. Rev Econ Stud **56**, 477–498 (1989)
Shneyerov, A., Wong, A.C.L.: The rate of convergence to perfect competition of matching and bargaining mechanisms. J Econ Theor **145**, 1164–1187 (2010)
Virág, G.: Competing auctions: Finite markets and convergence. Theor Econ **5**, 241–274 (2010)
Williams, S.R.: A characterization of efficient, Bayesian incentive compatible mechanisms. Econ Theor **14**, 155–180 (1999)
Wooders, J.: Matching and bargaining models of markets: approximating small markets by large markets. Econ Theor **11**, 215–224 (1998)
Zeithammer, R.: Commitment in sequential auctioning: advanced listings and threshold prices. Econ Theor **38**, 187–216 (2009)