# Uniform Price and Discriminatory Auctions with Variable Supply

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**Abstract**

This paper studies a divisible good auction model in which the seller determines supply after observing bids so as to maximize profit. Bidders have possibly asymmetric downward sloping demand curves and the seller has strictly increasing marginal cost. We show that, in all equilibria of the discriminatory auction, the Walrasian quantities are traded at the Walrasian price. Price discrimination does not materialize in equilibrium because bidders strategically misrepresent their demand, yet this feature eliminates the underpricing equilibria. In the uniform price auction, low-price equilibria exist even when supply is adjustable.

*Key Words:* variable supply auctions, discriminatory and uniform price auctions, subgame perfect equilibria
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# 1 Introduction

Auctions with variable supply are divisible good auctions in which the seller determines the quantity to be sold after the bidding in view of the bids received. Variable supply auctions are popular in many markets, e.g. markets for initial public offerings[1] and electricity,[2] but they are most widely used in Treasury markets.[3] For instance, Nyborg et al. (2002) point out that many Treasury departments have a policy of either rejecting bids or changing the amount to be sold in their Treasury auctions, and suggest that this policy has an impact on bidding. How do rational bidders behave in an auction in which the seller controls supply ex post? Should the seller use a uniform price or a discriminatory auction if supply is adjustable? And how efficient will the final allocation be?

These issues are still largely underresearched. The existing literature analyzed primarily the uniform price auction and explored whether the right to adjust supply eliminates the low-price equilibria known to exist in fixed supply models.[4] Back and Zender (2001) analyzed a Treasury auction model in which the seller has the right to reduce supply from a pre-specified level, and provided a lower bound on the market-clearing price. Thus, the seller's right to strategically withhold supply was found to reduce underpricing to a certain extent. McAdams (2007) developed this idea further and showed that, if the seller has increasing marginal cost, and can both reduce and extend supply, underpricing in the uniform price auction is entirely eliminated. Both papers rely on the assumption that all bidders have identical perfectly elastic demand curves, and discuss only the uniform price auction.

This paper, in contrast, analyzes both the uniform price and the discriminatory auction and considers a more general setting in which bidders have possibly asymmetric (weakly)

[1]For instance, in the IPO markets in the United States, the issuing firm can withdraw part or the entire issue if it considers the bids unsatisfactory. Busaba, Benveniste and Guo (2001), Brisley and Busaba (2003), and Dunbar and Foerster (2002) describe incidences of IPO withdrawals from the public press and provide statistics on withdrawn issues over the time period 1984-2000. Conversely, if demand is high, the issuing firm can exercise the so called over-allotment or "Greenshoe" option. This option allows the underwriter to sell up to an additional 15% of the issue, if the IPO is oversubscribed. Extensions and reductions of the offering are ubiquitous also in European IPOs.

[2]Loxley and Salant (2004, p.224) describes such an auction procedure used to allocate default service electricity in New Jersey in 2002.

[3]The Treasuries of Switzerland, Mexico, Italy, Sweden, Finland, Germany and Norway adjust the amount to be sold in their regular auctions after receiving bids. Heller and Lengwiler (2001) comment that the Swiss Treasury announces a maximum number of bonds that will be issued, but usually a significantly lower number is sold in the auction. Similar practice is reported by Umlauf (1993, pp. 316–317) regarding the Mexican Treasury and by Scalia (1997) with regard to the Treasury auctions in Italy. Keloharju, Nyborg and Rydqvist (2005) emphasize that the Finnish Treasury acts strategically by determining supply after observing bids; this type of behavior is also documented by Rocholl (2004) for the Treasury auctions in Germany, by Nyborg, Rydqvist and Sundaresan (2002) regarding the Swedish Treasury auctions, and by Bjonnes (2001) with respect to the Norwegian Treasury Bond auctions.

[4]Wilson (1979) first pointed out that the uniform price auction has low-price equilibria; see also Back and Zender (1993).

downward sloping demand curves. Similarly to the aforementioned literature, we assume that bidders are perfectly informed about other bidders' demand and the marginal cost of the seller. The paper provides two new insights on bidding in variable supply auctions.

First, we demonstrate that the discriminatory auction has a unique equilibrium outcome in which the Walrasian quantities are traded at the Walrasian price. Price discrimination does not materialize in equilibrium because bidders strategically misrepresent their true demand. They quote the same (stop-out) price and effectively counterbalance the discriminatory power of the monopolist. Yet, this strategic feature eliminates the underpricing equilibria in the auction. In a model of fixed supply and perfectly elastic demand Back and Zender (1993) demonstrate that bidders in the discriminatory auction will bid a price equal to the value of the good, $v$, which is assumed common knowledge. We show that more general result holds when supply is adjustable.

Second, we construct an equilibrium example of the uniform price auction in which the stop-out price is below the Walrasian price. Thus, the results relying on the assumption of perfectly elastic demand should be applied with care because they do not extend to a more general formulation of the market game.

Our results establish a ranking of the discriminatory and the uniform price auction in a variable supply model. In practice, both auctions are used. The Treasury departments of Sweden, Switzerland, Finland, Germany and Mexico[5] use a discriminatory auction; the Treasuries of Italy and Norway employ a uniform price auction. In all these countries the Treasuries determine supply ex post. The only variable supply model which discusses both the uniform price and the discriminatory auction is Lengwiler (1999). In this model the seller is privately informed about his constant marginal cost. Because of this assumption in the discriminatory auction bidders virtually do not compete: all bids above the marginal cost are served. This is in contrast to the analysis presented here. Additionally, Lengwiler restricts bidders' choice to the announcement of demand quantities only at two exogenously given price levels (we allow for general left-continuous bid functions), and takes no reference to the Walrasian price. This makes results incomparable, suggests, however, that the assumption of perfectly informed bidders is consequential for the results (ranking in Lengwiler's model is ambiguous).

## 2 Model and results

A monopolist auctions off a perfectly divisible good to $n \geq 2$ buyers. Each bidder $i$ has continuous and monotonically decreasing demand function $d_i(p)$. The seller has continuous, monotonically increasing and unbounded above marginal cost function $MC(Q)$.

[5] Umlauf (1993) reports that Mexico switched from a uniform price to a discriminatory auction in 1990.

That is, at price $p$, if acting as price taker, the seller will want to sell the amount $S(p) = MC^{-1}(p)$. Let us denote the aggregate demand function by $D(p) = \sum_{i=1}^{n} d_i(p)$, and the Walrasian price by $p^w$: $D(p^w) = S(p^w)$. We assume that two or more buyers demand positive quantities at that price. Bidders simultaneously announce non-increasing left-continuous demand schedules (bids) to the auctioneer. A bid of bidder $i$ is denoted by $b_i : [0, \infty) \to [0, \infty)$, and a bid vector of all bids is denoted by $b = (b_1, b_2, \ldots, b_n)$. After observing the bids and constructing the announced aggregate demand function, $B(p) = \sum_{i=1}^{n} b_i(p)$, the seller chooses actual supply, $Q$, so as to maximize his profit. Thus, seller's strategy, $\phi^*(\cdot)$, is a mapping from the set of all bid vectors, which we denote by $\mathfrak{Q}^n$, into a supply quantity $Q \geq 0$. Bids are served according to the "pro rata on the margin" rule.[6] The stop-out (or market-clearing) price in the auction is established as the highest price at which announced demand equals or exceeds supply: $p_S = \max\{p \mid B(p) \geq Q\}$.[7] Demand above the stop-out price is awarded in full, and the remaining supply is pro rated among the bidders who announced quantities at the stop-out price. Let us divide the announced demand of each bidder, $b_i(p_S)$, into quantity announced at prices higher than the stop-out price, $b_i^+(p_S) = \lim_{p \downarrow p_S} b_i(p)$, and remaining quantity announced at the stop-out price, $b_i^0(p_S) = b_i(p_S) - b_i^+(p_S)$ (flat portion or discontinuity in the announced demand curve). Bidder $i$ is awarded the quantity

$$Q_i = b_i^+(p_S) + \frac{b_i^0(p_S)}{\sum_{i=1}^{n} b_i^0(p_S)}(Q - \sum_{i=1}^{n} b_i^+(p_S)).$$

We will focus on the subgame-perfect equilibria of these two-stage market games. Let the payoff of the seller be denoted by $R(b_1, \ldots, b_n; \phi)$, and the payoff of bidder $i$ by $V_i(b_1, \ldots, b_n; \phi)$.

**Definition 1.** *The bid profile $b^* = (b_1^*, b_2^*, \ldots, b_n^*)$ and the seller's supply function $\phi^* : \mathfrak{Q}^n \to \mathbb{R}_+$ constitute a subgame perfect equilibrium, if they satisfy the following conditions. Second stage: The seller chooses a profit maximizing quantity for each bid vector,*

$$R(b; \phi^*(b)) \geq R(b; Q), \quad \forall Q \in \mathbb{R}_+ \quad \text{and} \quad \forall b \in \mathfrak{Q}^n.$$

*First stage: Bidders play a Nash equilibrium at the bidding stage of the game,*

$$V_i(b_i^*, b_{-i}^*; \phi^*(b^*)) \geq V_i(b_i, b_{-i}^*; \phi^*(b_i, b_{-i}^*)), \quad \forall b_i \in \mathfrak{Q}.$$

[6]This allocation rule gives a priority to high demand, and is the standard rule used in Treasury auctions. Roughly speaking, according to this rule bids are serviced in the order of descending price until supply is exhausted. See Kremer and Nyborg (2004) for a classification of rationing rules.

[7]If $B(0) < Q$ we assume that the stop-out price is zero.

With these preliminaries we characterize equilibrium bidding in the uniform price and the discriminatory auction.

**The discriminatory auction**

In the discriminatory auction each bidder pays the area under his announced demand curve. Formally, his total payment is $p_S \cdot Q_i + \int_{p_S}^{\infty} b_i(p)\mathrm{d}p$. The next theorem establishes our main result.

**Theorem 1.** *In all (subgame perfect) equilibria of the discriminatory auction the Walrasian quantities are traded at the Walrasian price. Bidders submit flat demand curves at the Walrasian price, proportionally overstating their actual demand:*

$$b_i^*(p) = \begin{cases} 0 & \textit{for } p > p^w, \\ \lambda \cdot d_i(p^w) & \textit{for } p \le p^w, \end{cases}$$

*where*

$$\lambda \ge \frac{D(p^w)}{D(p^w) - \max\limits_{i \in \{1,2,\ldots,n\}} \{d_i(p^w)\}}.$$

In equilibrium, bidders overstate their true demand at the Walrasian price by a common factor $\lambda$. The role of the threshold value $\frac{D(p^w)}{D(p^w) - \max\limits_{i \in \{1,2,\ldots,n\}} \{d_i(p^w)\}}$ is to ensure that no bidder has a deviation that can lower the stop-out price. Indeed, even if one of the bidders lowers his bid, the announced aggregate demand of the other bidders will correspond to the Walrasian quantity or be higher. In such a case the seller will not serve the deviating bidder, and such a deviation will not be profitable. Next, we present a formal proof of the theorem.

*Proof.* The proof proceeds by contradiction and is organized in four steps. Let us denote the equilibrium stop-out price by $p_S^*$. In the discriminatory auction the seller acts as a perfectly discriminating monopolist with respect to the received bids, so at the stop-out price he will supply $\min\{B(p_S^*), S(p_S^*)\}$.

*Step 1. All bidders announce flat demand curves at the same (stop-out) price $p_S^*$.*

Assume that there exists a bidder $i$ who does not submit (entirely) flat demand at the stop-price. Let the quantity announced at higher prices than the stop-out price be $b_i^{*+}(p_S^*)$. This quantity will be awarded in full, and the bidder will be charged $p_S^* \cdot b_i^{*+}(p_S^*) + \int_{p_S}^{\infty} b_i^*(p)\mathrm{d}p$

for it, where by assumption $\int\limits_{p_S}^{\infty} b_i^*(p)\mathrm{d}p > 0$. Consider the following deviation of that bidder:

$$b_i^\epsilon(p) = \begin{cases} 0 & \text{for } p > p_S^* + \epsilon, \\ b_i^{*+}(p_S^*) & \text{for } p_S^* + \epsilon \geq p > p_S^*, \\ b_i^*(p_S^*) & \text{for } p \leq p_S^*, \end{cases}$$

where $\epsilon > 0$ is a small number. The seller will not change supply as a result of this deviation, and that bidder will receive the same quantity as before. For the quantity $b_i^{*+}(p_S^*)$ the new amount to be paid will be $(p_S^* + \epsilon) \cdot b_i^{*+}(p_S^*) = p_S^* \cdot b_i^{*+}(p_S^*) + \epsilon \cdot b_i^{*+}(p_S^*)$. For a sufficiently small $\epsilon$ we obtain $\epsilon \cdot b_i^{*+}(p_S^*) < \int\limits_{p_S}^{\infty} b_i^*(p)\mathrm{d}p$. The deviation is profitable, a contradiction. Therefore, from now on we will consider only flat bids submitted at the same (stop-out) price.

*Step 2. The stop-out price is not lower than the Walrasian price.*

Assume $p_S^* < p^w$. The seller will supply no more than $S(p_S^*)$ which is lower than aggregate demand $D(p_S^*)$. This implies that at least one bidder obtains an amount lower than his actual demand at that price. In this case announcing a slightly higher quantity at the same price $p_S^*$ will allow this bidder to obtain a slightly higher quantity and such a deviation will be profitable.

*Step 3. The stop-out price is not higher than the Walrasian price.*

Assume $p_S^* > p^w$. If such an equilibrium exists, each bidder $i$ will be awarded the quantity $d_i(p_S^*)$. Indeed, if a bidder receives more (or less) than $d_i(p_S^*)$, he can reduce (extend) his bid announcement at the price $p_S^*$ accordingly so as to obtain the amount $d_i(p_S^*)$. With this argument we conclude that announced aggregate demand will be

$$B(p) = \begin{cases} 0 & \text{for } p > p_S^*, \\ D(p_S^*) & \text{for } p \leq p_S^*. \end{cases}$$

Consider now the following deviation of bidder $i$:

$$b_i^\epsilon(p) = \begin{cases} 0 & \text{for } p > p_S^* - \epsilon, \\ b_i^*(p_S^*) & \text{for } p \leq p_S^* - \epsilon. \end{cases}$$

We have $D(p_S^*) < S(p_S^*)$, and if $p_S^*$ is reduced by a small enough $\epsilon > 0$, the inequality will not change sign, i.e. $D(p_S^* - \epsilon) < S(p_S^* - \epsilon)$. Hence, for small enough $\epsilon > 0$, the seller will lower the stop-out price to serve the deviating bidder. Bidder $i$ will obtain the same quantity, $d_i(p_S^*)$, at a unit price of $(p_S^* - \epsilon)$. The deviation is profitable.

*Step 4. The strategy profiles, as formulated in the Theorem, are equilibrium profiles. In every equilibrium the Walrasian quantities are traded at the Walrasian price.*

It is easily observed that no profitable deviations exist. Since all bidders overstate their demand, if a bidder lowers his bid, he will not be served because the seller can sell the quantity $S(p^w)$ at the price $p^w$.[8] If a bidder deviates from the quantity at $p^w$ as stated in the Theorem, he will be awarded a quantity different than $d_i(p^w)$, which is unprofitable. Therefore, no other equilibria exist. □

**The uniform price auction**

In the uniform price auction all bidders pay the stop-out price for each awarded unit. The equilibrium strategy profile stated in Theorem 1 is easily seen to be an equilibrium also in the uniform price auction. The next example illustrates an underpricing equilibrium.

**Example 1.** *Two bidders participate in a uniform price auction. The demand of each bidder $i \in \{1, 2\}$ is*

$$d_i(p) = \begin{cases} 0 & \text{for } p > 2.2, \\ 1 & \text{for } p \leq 2.2. \end{cases}$$

*The marginal cost of the seller is $MC(Q) = Q$. The Walrasian price is $p^w = 2$. The bids*

$$b_1^*(p) = \begin{cases} 0 & \text{for } p > 2, \\ 1 & \text{for } p \leq 2, \end{cases}$$

*and*

$$b_2^*(p) = \begin{cases} 0 & \text{for } p > \sqrt{3}, \\ \sqrt{3} & \text{for } p \leq \sqrt{3}. \end{cases}$$

*are equilibrium bids. The seller supplies the quantity $\sqrt{3}$, the stop-out price is $\sqrt{3}$, and the bidders are granted the quantities $Q_1^* = 1$, $Q_2^* = \sqrt{3} - 1$.*

Quantities 1 and $\sqrt{3}$ are optimal for the seller. They generate a profit of $\frac{3}{2}$, so we consider the case in which supply is $\sqrt{3}$. Bidder 1 obtains his desired quantity at the stop-out price $\sqrt{3}$. He has no deviation which can lower the stop-out price because, if he submits a bid price below $\sqrt{3}$, he will not be served. In such a case the auctioneer will sell the quantity $\sqrt{3}$ to bidder 2. Bidder 1 obtains his desired quantity, so deviations that lead to different quantities are also not profitable. Observe that in the case of perfectly elastic and unrestricted demand this argument unravels, and low-price equilibria

[8] Observe that if all bidders announce their demand truthfully, then a bidder who slightly lowers his bid price might still be served, and the benefit from the reduction in price might overcompensate the reduction in quantity. So, such a deviation might be profitable. Proportional overstatement of demand eliminates this possibility.

do not exist.[9] Bidder 2 has no profitable deviations either. If he submits a bid price lower than $\sqrt{3}$ he will not be served. The profit of the monopolist will drop below $\frac{3}{2}$ if he serves both bidders, and he will prefer to sell only one unit to bidder 1 at the price of 2. Submitting a higher price is also unprofitable for bidder 2. Indeed, if he submits a price of $p_2 > \sqrt{3}$ he will obtain a quantity not larger than $(p_2 - 1)$. His payoff then will be $(2.2-p_2)(p_2-1)$. This is lower than the profit at the price $\sqrt{3}$, which is $(2.2-\sqrt{3})\cdot(\sqrt{3}-1)$.

The Theorem and the Example provide a ranking of the two auctions. While the outcome of the discriminatory auction is essentially unique and corresponds to the Walrasian allocation, the uniform price auction admits a variety of allocations. Along with the equilibria that produce the Walrasian allocation, there is a plethora of other equilibria. As the Example demonstrates, in these equilibria the price is lower, and lower quantities are traded. The seller's option to control supply after the bidding does not seem alone to be able to remedy the low-price equilibrium problem inherent to the uniform-price auction. To eliminate the low-price equilibria, the seller can use special rationing rules, e.g. the pro-rata rationing rule (see Damianov 2005). As we demonstrate here, the seller can alternatively use a discriminatory auction, and this auction has no low-price equilibria even when the traditional pro-rata on the margin rationing rule is used.

# 3 Conclusion

This paper presents an equilibrium bidding model of a variable supply auction. Our model accounts for the divisibility of the good, the complexity of bidders' strategy space, and discusses both the uniform price and the discriminatory price auctions. A major limitation of the literature on the topic (including this model) is the assumption of complete information regarding bidders' demand curves. This assumption is in contrast to single-unit auction theory, where the focus is on incomplete information. It allows us to explore important new issues concerned with the divisibility of the good and the endogeneity of supply. Relaxing this assumption would be the next step toward a more thorough understanding of the allocation properties of these important trade mechanisms.

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[9] Another way to resolve the low-price equilibrium problem is to apply the "pro rata" rationing rule (see Damianov 2005). However, this allocation rule might be problematic in practice because it does not give priority to high demand. That is, bids at lower prices are serviced even when bids at higher prices are not completely filled.

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